CALCULUS 1

UNIT 3

THE DERIVATIVE

LESSON 5

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THE CHAIN RULE

The **chain rule** provides us with a technique for finding the derivative of composite functions, with the number of functions that make up the composition determining how many differentiation steps are necessary. For example, if a composite function *f*(x) is defined as

$$f(x) = (g \circ h)(x) = g[h(x)]$$

$$then, f'(x) = g'[h(x)] * h'(x)$$

Note that because two functions, *g,* and *h*, make up the composite function *f*, you must consider the derivatives *g′* and *h′* in differentiating *f*(x).

Chain Rule Steps

- Step 1: Identify The Chain Rule: The function must be a composite function, which means one function is nested over the other.
- Step 2: Identify the inner function and the outer function.
- Step 3: Find the derivative of the outer function, leaving the inner function.
- Step 4: Find the derivative of the inner function.
- Step 5: Multiply the results from step 4 and step 5.
- Step 6: Simplify the chain rule derivative.

Worked Examples.

Differentiate the following.

Example 1

$$f(x) = 5(x^2 + 3)^3$$

Solution

$$f^1(x) = 15(x^2 + 3)^2(2x)$$

$$= 30x(x^2 + 3)^2$$

Example 2

$$f(x) = (3x^2 + 5x - 1)$$

Solution

$$f^1(x) = 40(3x^2 + 5x - 1)(6x + 5)$$

Example 3

$$f(x) = \sin(2x)^4$$

Solution

$$f^1(x) = \sec(2x)^4 * \tan(2x)^4(8x^3)$$

$$= 8x^3 \sec(2x)^4 * \tan(2x)^4$$

Example 4

$$f^1(x) = (3x^3 + 1)(-4x^2 - 3)^4$$

Note**:** To solve this problem, we must use both the product rule and the chain rule.

Solution

$$f'(x) = (3x^3+1)(4)(-4x^2-3)^3(-8x) + (-4x^2-3)^4(9x^2)$$
$$= (-4x^2-3)^3\{(3x^3+1)(-8x) + (9x^2)(-4x^2-3)\}$$
$$= (-4x^2-3)^3\{4(-24x^4+8x) + (-36x^3-27x^2)\}$$
$$= (-4x^2-3)^3\{(-96x^4+32x) + (-36x^3-27x^2)\}$$
$$= (-4x^2-3)^3\{x(-96x^3+32) - x(36x^3+27x)\}$$
$$= x(-4x^2-3)^3(-132x^3-27x+32)$$

Exercise

1) $y = (5x^4+1)^2$
2) $f(x) = (6x^3-5)^4$
3) $g(x) = (-5x^3-3)^3$
4) $y = (4x^2+2)^4$
5) $h(x) = (-7x^4+3)^7$
6) $f(x) = (x^4+2)^3$
7) $g(x) = (3x-1)(-3x^3-4)^{-3}$
8) $y = (x^3-4x^2+5)^6$
9) $h(x) = \dfrac{(x^3+4)^5}{3x^4-2}$
10) $f(x) = \left(\dfrac{5x^5-3}{-3x^3+1}\right)^3$

Solution

1) $y' = 2(5x^4+1)(20x^3)$

$= 40x^3(5x^4+1)$

2) $f'(x) = 4(6x^3-5)^3(18x)$

$= 72x(6x^3-5)^3$

3) $g'(x) = 3(-5x^3-3)^2(-15x^2)$

$= -45x^2(-5x^3-3)^2$

4) $y' = 4(4x^2+2)^3(8x)$

$= 32x(4x^2+2)^3$

5) $h'(x) = 7(-7x^4+3)^6(-28x^3)$

$= -196x^3(-7x^4+3)^6$

6) $f'(x) = 3(x^4+2)^2(4x^3)$

$= 12x^3(x^4+2)^2$

7)

$$g'(x) = (3x-1)(-3)\left(-3x^2-4\right)^{-4}(-6x) + \left(-3x^2-4\right)^{-3}(3)$$
$$= \left(-3x^2-4\right)^{-4}\{18x(3x-1) + 3\left(-3x^2-4\right)^{-3}\}$$
$$= \frac{54x^2-18x-9x^2-12}{\left(-3x^2-4\right)^4}$$
$$= \frac{45x^2-18x-12}{\left(-3x^2-4\right)^4}$$
$$= \frac{3\left(15x^2-6x-4\right)}{\left(-3x^2-4\right)^4}$$

8) $y' = 6\left(x^3-4x^2+5\right)\left(3x^2-8\right)$

9)

$$h'(x) = \frac{\left(3x^4-2\right)(5)\left(x^3+4\right)^4\left(3x^2\right) - \left(x^3+4\right)^5\left(12x^3\right)}{\left(3x^4-2\right)^2}$$
$$= \frac{\left(x^3+4\right)^4\{15x^2\left(3x^4-2\right) - 12x^3\left(x^3+4\right)\}}{\left(3x^4-2\right)^2}$$
$$= \frac{\left(x^3+4\right)^4\{\left(45x^6-30x^2\right) - \left(12x^6+48x^3\right)\}}{\left(3x^4-2\right)^2}$$
$$= \frac{\left(x^3+4\right)^4\left(33x^6-48x^3-30x^2\right)}{\left(3x^4-2\right)^2}$$
$$= \frac{3x^2\left(x^3+4\right)^4\left(11x^4-16x-10\right)}{\left(3x^4-2\right)^2}$$

10)

$$f'(x) = 3\left\{\frac{5x^5-3}{-3x^3+1}\right\}^2\left\{\frac{\left(-3x^3+1\right)25x^4 - \left(5x^5-3\right) * -9x^2}{\left(-3x^3+1\right)^2}\right\}$$
$$= 3\left\{\frac{5x^5-3}{-3x^3+1}\right\}^2\left\{\frac{\left(-75x^7+25x^4+45x^7-27x^2\right)}{\left(-3x^3+1\right)^2}\right\}$$
$$= \frac{3x^2\left(5x^5-3\right)^2\left(-30x^5+25x^2-27\right)}{\left(-3x^3+1\right)4}$$