CALCULUS 1

UNIT 3

THE DERIVATIVE

LESSON 1

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WHAT IS THE DERIVATIVE

The **derivative** of a function of a single variable at a chosen input value, when it exists, is the slope of the tangent line to the graph of the function at that point. In other words, the derivative is the instantaneous rate of change of a function with respect to one of its variables. As was indicated above, it is the equivalent to finding the slope of the tangent line at a given point.

Derivative Notations

Suppose you have a general function: $y = f(x)$. All the following notations can be read as "the derivative of y with respect to x" or less formally, "the derivative of the function."

$f'(x)$ f' y' df/dx dy/dx $d/dx\,[f(x)]$.

The Difference Quotient

We use many formulas in mathematics and one of these formulas, called the Difference Quotient, is used to solve for the derivative of a function.

The formula is:

$$f'(x) = \frac{f(x+h) - (x)}{h}$$

The name of the formula is based on the two operators that are in the equation. There is a fraction with a subtraction (the difference) in the numerator. Hence the name Difference Quotient. We have both a subtraction and a division (numerator divided by denominator). The symbols may seem confusing, but this will be discussed very shortly.

Note the following:

I. The ability to set up and simplify difference quotients is essential for calculus students. It is from the difference quotient that the elementary formulas for derivatives are developed.

II. Setting up a difference quotient for a given function requires an understanding of function notation.

III. Given the function: f(x) = 3x^2 - 4x- 5

A. This notation is read “f of x equals . . .”.

B. The implication is that the value of the function (the y-value) depends upon the replacement for “x”.

C. If a number is substituted for “x”, a numerical value for the function is found.

D. If a non-numerical quantity is substituted for “x”, an expression is found rather than a numerical value.

E. Careful use of parentheses is essential!

Example 1

Example 2

$f(x) = x^2 + 4x - 6$

$$\frac{f(x+h) - f(h)}{h} = \frac{(x+h)^2 + 4(x+h) - 6 - (x^2 + 4x - 6)}{h}$$

$$\frac{x^2 + 2xh + h^2 + 4x + 4h - 6 - x^2 - 4x + 6}{h}$$

$$\frac{2xh + h^2 + 4h}{h} = \frac{h(2x + h + 4)}{h} = 2x + h + 4$$

Example 3

$f(x) = -16x^2 + 9x + 8$

$$\frac{f(x+h) - f(x)}{h} = \frac{-16(x+h)^2 + 9(x+h) + 8 - (-16x^2 + 9x + 8)}{h}$$

$$= \frac{-16(x^2 + 2xh + h^2) + 9x + 9h + 8 + 16x^2 - 9x - 8}{h}$$

$$= \frac{-16x^2 - 32xh - 16h^2 + 9x + 9h + 8 + 16x^2 - 9x - 8}{h}$$

$$= \frac{-32xh - 16h^2 + 9h}{h}$$

$$= \frac{h(-32x - 16h + 9)}{h}$$

$$= -32x - 16h + 9$$

Example 4

$f(x) = \frac{1}{2}x^2 + 5x + 3$

$$\frac{f(x+h) - f(h)}{h} = \frac{\frac{1}{2}(x+h)^2 + 5(x+h) + 3 - \left(\frac{1}{2}x^2 + 5x + 3\right)}{h}$$

$$= \frac{\frac{1}{2}(x^2 + 2xh + h^2) + 5x + 5h + 3 - \frac{1}{2}x^2 - 5x - 3}{h}$$

$$= \frac{\frac{1}{2}x^2 + xh + \frac{1}{2}h^2 + 5x + 5h + 3 - \frac{1}{2}x^2 - 5x - 3}{h}$$

$$= \frac{xh + \frac{1}{2}h^2 + 5h}{h} = \frac{h\left(x + \frac{1}{2}h + 5\right)}{h} = x + \frac{1}{2}h + 5$$

Lesson 1 Exercise

1) $f(x) = x^2 - 5x - 1$
2) $f(x) = 5x - 2x^2$
3) $f(x) = 17x^2 - 13x$
4) $f(x) = 3x^2 - 11x - 7$

5) $f(x)=2x^2-3x+2$

6) $f(x)=x^2-2x-6$

7) $f(x)=5x^2-6x+1$

8) $f(x)=2x^2-7x-5$

9) $f(x)=12x^2+9x+10$

10) $f(x)=2x^2+5x-1$

SOLUTIONS

1) $f(x)=x^2-5x-1$

$$\frac{f(x+h)-f(x)}{h}$$

$$=\frac{\left(x^2+2xh+h^2\right)-5x-5h-1-\left(x^2-5x-1\right)}{h}$$

$$=\frac{x^2+2xh+h^2-5x-5h-1-x^2+5x+1}{h}$$

$$=\frac{h(2x+h-5)}{h}=2x+h-5$$

2) $f(x)=5x-2x^2$

$$\frac{f(x+h)-f(x)}{h}$$

$$=\frac{5x+5h-2\left(x^2+2xh+h^2\right)-\left(5x-2x^2\right)}{h}$$

$$=\frac{5x+5h-2x^2-4xh+2h^2-5x+2x^2}{h}$$

$$=\frac{5h-4xh+2h^2}{h}=\frac{h(5-4x+2h)}{h}=5-4x+2h$$

3) $f(x)=17x^2-13x$

$$\frac{f(x+h)-f(x)}{h}$$

$$=\frac{17\left(x^2+2xh+h^2\right)-13x-13h-\left(17x^2-13x\right)}{h}$$

$$=\frac{17x^2+34xh+17h^2-13x-13h-17x^2+13x}{h}$$

$$=\frac{34xh+17h^2-13h}{h}=\frac{h(34x+17h-13)}{h}=34x+17h-13$$

4) $f(x)=3x^2-11x-7$

$$\frac{f(x+h)-f(x)}{h}=\frac{3(x+h)^2-11(x+h)-7-\left(3x^2-11x-7\right)}{h}$$

$$=\frac{3\left(x^2+2xh+h^2\right)-11(x+h)-7-3x^2+11x+7}{h}$$

$$=\frac{3x^2+6xh+3h^2-11x-11h-7-3x^2+11x+7}{h}$$

$$=\frac{6xh+3h^2-11h}{h}=\frac{h(6x+3h-11)}{h}=6x+3h-11$$

5) $f(x)=2x^2-3x+2$

$$\frac{f(x+h)-f(x)}{h}=\frac{2(x+h)^2-3(x+h)+2-\left(2x^2-3x+2\right)}{h}$$

$$=\frac{2(x+h)^2-3(x+h)+2-\left(2x^2-3x+2\right)}{h}$$

$$=\frac{2\left(x^2+2xh+h^2\right)-3x-3h+2-2x^2+3x-2}{h}$$

$$=\frac{2x^2+4xh+2h^2-3x-3h+2-2x^2+3x-2}{h}$$

$$=\frac{4xh+2h^2-3h}{h}=\frac{h(4x+2h-3)}{h}=4x+2h-3$$

6) $f(x)=x^2-2x-6$

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-2(x+h)-6-\left(x^2-2x-6\right)}{h}$$

$$=\frac{(x+h)^2-2(x+h)-6-\left(x^2-2x-6\right)}{h}$$

$$=\frac{x^2+2xh+h^2-2x-2h-6-x^2+2x+6}{h}$$

$$=\frac{2xh+h^2-2h}{h}=\frac{h(2x+h-2)}{h}=2x+h-2$$

7) $f(x)=5x^2-6x+1$

$$\frac{f(x+h)-f(x)}{h}=\frac{5(x+h)^2-6(x+h)+1-\left(5x^2-6x+1\right)}{h}$$

$$=\frac{5\left(x^2+2xh+h^2\right)-6(x+h)+1-5x^2+6x-1}{h}$$

$$=\frac{5x^2+10xh+5h^2-6x-6h+1-5x^2+6x-1}{h}$$

$$=\frac{10xh+5h^2-6h}{h}=\frac{h(10x+h-6)}{h}=10x+h-6$$

8) $f(x)=2x^2-7x-5$

$$\frac{f(x+h)-f(x)}{h}=\frac{2(x+h)^2-7(x+h)-5-\left(2x^2-7x-5\right)}{h}$$

$$=\frac{2\left(x^2+2xh+h^2\right)-7(x+h)-5-2x^2+7x+5}{h}$$

$$=\frac{2x^2+4xh+2h^2-7x-7h-2x^2+7x+5}{h}$$

$$=\frac{4xh+2h^2-7h}{h}=\frac{h(4x+2h-7)}{h}=4x+2h-7$$

9) $f(x)=12x^2+9x+10$

$$\frac{f(x+h)-f(x)}{h}=\frac{12(x+h)^2+9(x+h)+10-\left(12x^2+9x+10\right)}{h}$$

$$=\frac{12\left(x^2+2xh+h^2\right)+9(x+h)+10-12x^2-9x-10}{h}$$

$$=\frac{12x^2+24xh+12h^2+9x+9h+10-12x^2-9x-10}{h}$$

$$=\frac{24xh+12h^2+9h}{h}=\frac{h(24x+12h+9)}{h}=24x+12h+9$$

10) $f(x)=2x^2+5x-1$

$$\frac{f(x+h)-f(x)}{h}=\frac{2(x+h)^2+5(x+h)-1-\left(2x^2+5x-1\right)}{h}$$

$$=\frac{2\left(x^2+2xh+h^2\right)+5(x+h)-1-2x^2-5x+1}{h}$$

$$=\frac{2x^2+4xh+2h^2+5x+5h-1-2x^2-5x+1}{h}$$

$$=\frac{4xh+2h^2+5h}{h}=\frac{h(4x+2h+5)}{h}=4x+4h+5$$