CALCULUS 1

UNIT 2

CONTINUITY

LESSON 1

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In calculus, a function is continuous at x = a, if and only if all three of the following conditions are met:

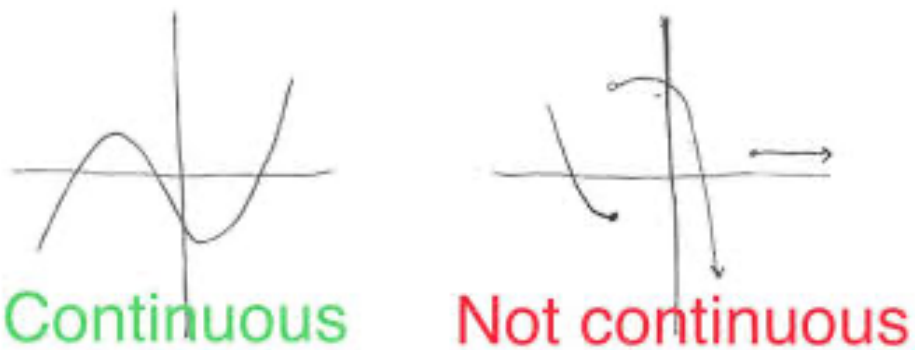
(a) The function is defined at $x = a$; that is, $f(a)$ equals a real number.

(b) The limit of the function as x approaches a exists.

(c) The limit of the function as x approaches a is equal to the function value at $x = a$.

Put simply, a continuous function is one that could be drawn without lifting your pencil from the paper, i.e., one with no gaps. Put another way, a function is said to be continuous on an interval if there are no breaks, no holes, or no jumps in that interval. (See figure below).

Remember that polynomial functions, exponential functions, and the sine and cosine functions are continuous on every interval. However, rational functions are only continuous on the intervals where their denominators are not zero.



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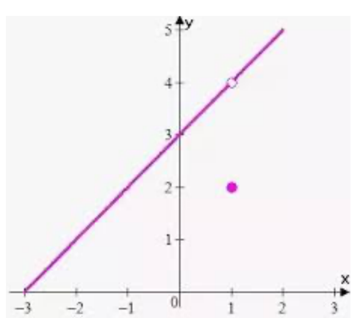
Note that Discontinuities may be Removable or Non-Removable.

Example of Removable Discontinuities.

A removable discontinuity is hole in a graph. That is, a discontinuity that can be "repaired" by filling in a single point. In other words, a removable discontinuity is a point at which a graph is not connected but can be made connected by filling in a single point. When graphed, a removable discontinuity is marked by an open circle on the graph at the point where the graph is undefined or is a different value. Removable discontinuities are also characterized by the fact that the limit exists and they can be repaired by re-defining the function.



How to solve a removable discontinuity.

Step 1: Factor the numerator and the denominator.
Step 2: Identify factors that occur in both the numerator and the denominator.
Step 3: Set the common factors equal to zero.
Step 4: Solve for x.

Example

$f(x) = \frac{x^3+27}{x+3}$; when x = -3.

$f(x) = \frac{x^3+27}{x+3} = \frac{-3^3+27}{-3+3} = \frac{-27+27}{-3+3} = \frac{0}{0}$ Undefined, then we must factor the numerator.

$$f(x) = \frac{x^3+27}{x+3} = \frac{(x+3)(x^2-3x+9)}{x+3}$$

$$f(x) = x^2-3x+9 = (-3)^2-3(-3)+9 = 9+9+9 = 27$$

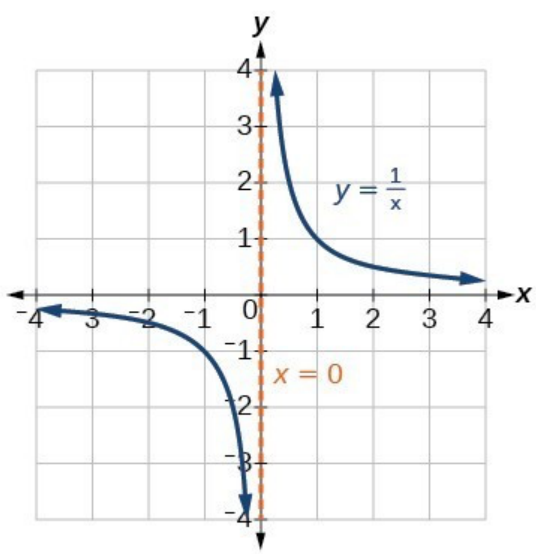
Therefore,

$f(x) = \frac{x^3+27}{x+3}$ is continuous when $x \neq -3$

Non-Removable Discontinuities

A non-removable discontinuity is a point at which a function is not continuous or is undefined and cannot be made continuous by being given a new value at the point. Consider the function $f(x) = 1/x$. (shown below). The function is not defined at $x = 0$. It cannot be extended to a continuous function since no matter what value is assigned at 0, the resulting function will not be continuous.

Example of a Non – Removable Discontinuity



Example 1.
Determine if the function is continuous. If it is not continuous, find the *x*-axis location of and classify each discontinuity.

$$f(x) = -\frac{x^2}{2x+4}$$

Solution

$2x + 4 = 0$ Non-Removable Discontinuity at $x = -2$.

$2x = -4$

$x = -2$

Example 2.
Determine if the function is continuous. If it is not continuous, find the *x*-axis location of and classify each discontinuity.

$$f(x) = -\frac{x+1}{x^2-x-2}$$

Solution

$x^2 - x - 2 = 0$ Removable Discontinuity at $x = -1$, Non-Removable Discontinuity at $x = 2$

$(x+1)(x-2) = 0$

$x = -1, 2$

Example 3.

$f(x) = \frac{|x-3|}{x-3}$ Non-Removable Discontinuity at $x = 3$

Lesson 1 Exercise

Determine if each function is continuous at the given *x*-value. If not continuous, classify each discontinuity.

1. $y = \frac{x+1}{|x+1|}$; at $x = 1$ and -1
2. $f(x) = \frac{x+2}{x^2-4}$; at $x = 2$ and $x = -2$
3. $f(x) = \frac{x^2}{x+1}$; at $x = -1$
4. $f(x) = \frac{x^2+4x+3}{x+3}$; at $x = 3$ and $x = -3$
5. $f(x) = \frac{x^2-x-2}{x+1}$; at $x = 1$ and $x = -1$
6. $f(x) = \frac{x-3}{x^2-x}$; at $x = 0$ and $x = 1$

Find the interval on which each function is continuous.

7. $f(x) = \frac{x-1}{x^2-x}$
8. $f(x) = \begin{cases} x^2 - 2x + 2, & x < 1 \\ 2x - 1, & x \geq 1 \end{cases}$
9. $f(x) = \begin{cases} x^2 + 2x + 1, & x < 1 \\ -\frac{1}{2}x, & \geq 1 \end{cases}$
10. $f(x) = \begin{cases} x^2 - 4x + 3, & x \neq 0 \\ 3, & x = 0 \end{cases}$

11. $f(x) = \begin{cases} 2x-10, x < 2 \\ 0, x \geq 2 \end{cases}$

12. $f(x) = \dfrac{x^2 - x - 12}{x + 3}$

13. $f(x) = \dfrac{x^2 - x - 6}{x + 2}$

14. $f(x) = \dfrac{x + 1}{x^2 + x + 1}$

15. $f(x) = \begin{cases} x, x < -1 \\ -x^2 + 2x, x \geq -1 \end{cases}$

SOLUTIONS

Determine if each function is continuous at the given *x*-value. If not continuous, classify each discontinuity.

1. $y = \dfrac{x+1}{|x+1|}$; at $x = 1$ and -1. Discontinuous at $x = -1$.
2. $f(x) = \dfrac{x+2}{x^2-4}$; at $x = 2$ and $x = -2$. Removable discontinue at $x = -2$, Infinite discontinuity at $x = 2$.
3. $f(x) = \dfrac{x^2}{x+1}$; at $x = -1$. Infinite Discontinuity at $x = -1$
4. $f(x) = \dfrac{x^2+4x+3}{x+3}$; at $x = 3$ and $x = -3$. Removable Discontinuity at $x = -3$
5. $f(x) = \dfrac{x^2-x-2}{x+1}$; at $x = 1$ and $x = -1$. Removable Discontinuity at $x = -1$
6. $f(x) = \dfrac{x-3}{x^2-x}$; at $x = 0$ and $x = 1$. Infinite Discontinuity at $x = 0, 1$

Find the interval on which each function is continuous.

7. $f(x) = \dfrac{x-1}{x^2-x}$. Answer: $(-\infty, 0), (0, 1), (1, \infty)$
8. $f(x) = \begin{cases} x^2 - 2x + 2, x < 1 \\ 2x-1, x \geq 1 \end{cases}$ Answer: $(-\infty, 0), (1, \infty)$
9. $f(x) = \begin{cases} x^2 + 2x + 1, x < 1 \\ -\dfrac{1}{2}x, \geq 1 \end{cases}$ Answer: $(-\infty, \infty), (1, \infty)$
10. $f(x) = \begin{cases} x^2 - 4x + 3, x \neq 0 \\ 3, x = 0 \end{cases}$ Answer: $(-\infty, \infty)$

11. $f(x)=\begin{cases}2x-10, x<2\\ 0, x\geq 2\end{cases}$ Answer: $(-\infty,\infty)$

12. $f(x)=\dfrac{x^2-x-12}{x+3}$ Answer: $(-\infty,\infty)$

13. $f(x)=\dfrac{x^2-x-6}{x+2}$ Answer: $(-\infty,\infty)$

14. $f(x)=\dfrac{x+1}{x^2+x+1}$ Answer $(-\infty,\infty)$

15. $f(x)=\begin{cases}x, x<-1\\ -x^2+2x, x\geq -1\end{cases}$ Answer: $(-\infty,-1),(-1,\infty)$