CALCULUS 1

UNIT 1

LIMITS

Lesson 1

LESSON 1

WHAT IS A LIMIT

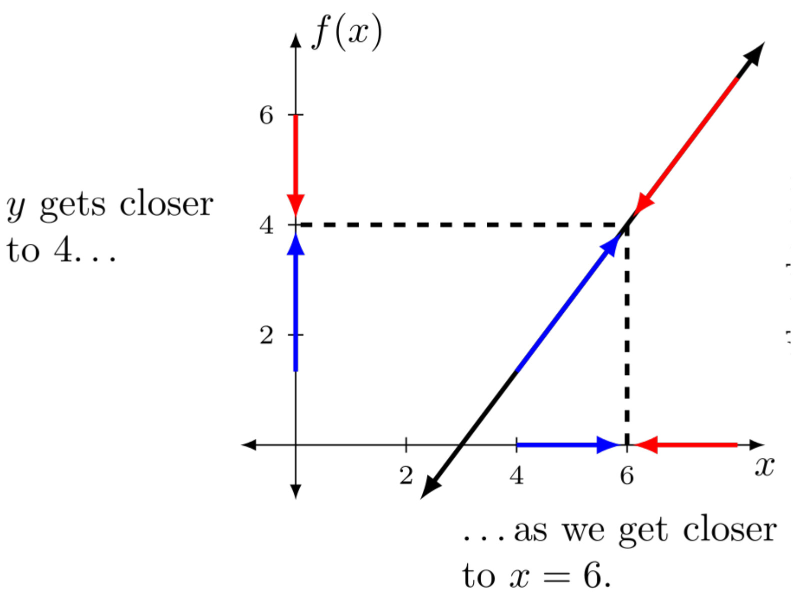
Put in very simple language a **limit** is the value that a function approaches as that function's inputs get closer and closer to some number.

According to Encyclopedia Britannica "Limits are the method by which the derivative, or rate of change, of a function is calculated, and they are used throughout analysis as a way of making approximations into exact quantities, as when the area inside a curved region is defined to be the limit of approximations by rectangles."

A more formal definition is:

Let I be an open interval containing c, and let *f* be a function defined on I, except possibly at c. The limit of *f*(*x*), as *x* approaches c, is L, denoted by $\boxed{\lim_{x\to c} f(x) = L}$

The figure below shows that as *x* approaches 6 from either side, *y* approaches 4. Note that we are not concerned about what occurs at 4 only very close to 4. We can say therefore that 4 is the limit of this function.



Limits that fail to exist

Example of a limit that does not exist.

Find $\lim_{x\to 0} \frac{1}{x}$.

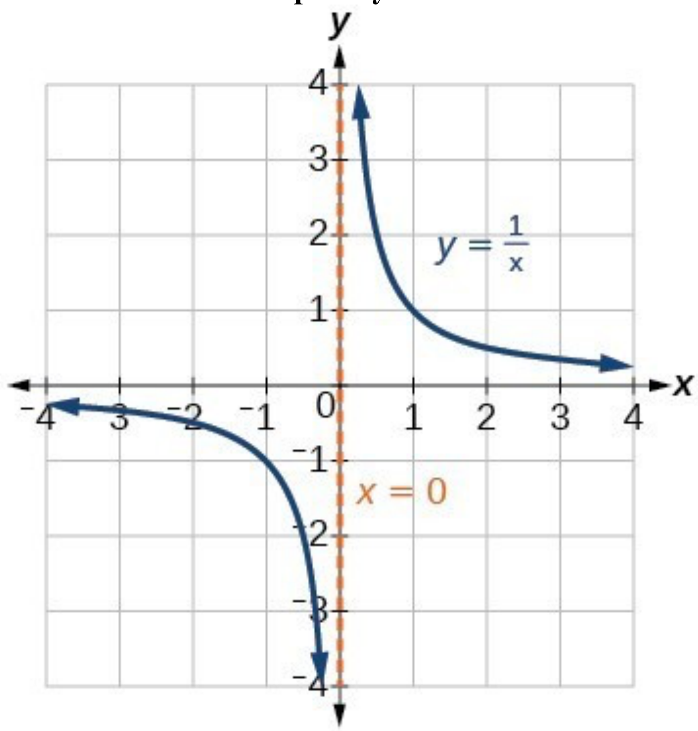
If we substitute zero in the expression, we will see that the limit does not exist since dividing by zero is undefined. However, this can be better demonstrated by trying to find the limit numerically. See table below.

From the left → ← From the right

x	**-0.5**	**-0.001**	**-0.0001**	**0**	**0.0001**	**0.001**	**0.5**
1/*x*	**-2**	**-1000**	**-10000**	**undef**	**10000**	**1000**	**2**

As can be seen from the table, as $x \to 0$ **from** the left, $1/x$ approaches negative infinity, and as $x \to 0$ **from** the right, $1/x$ approaches positive infinity. The graph of $y = 1/x$ is shown below.

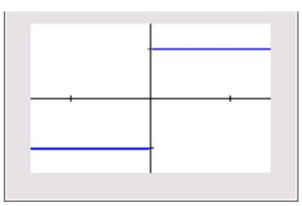
Graph of y = 1/*x*



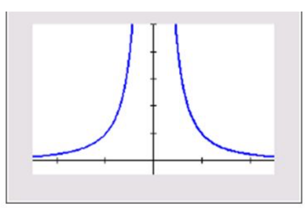
There are three main ways that limits fail to exist.

- The function approaches different values from the left and the right,
- The function grows without bound, and
- The function oscillates.

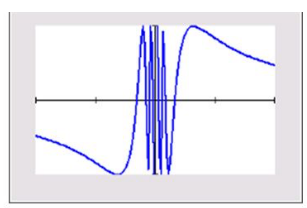
These three ways are shown below.



$\lim_{x\to 0} \frac{|x|}{x}$ does not exist (D.N.E. for short) because the behavior of the function differs from the left and the right.



$\lim_{x\to 0} \frac{1}{x^2}$ does not exist because the graph displays unbounded behavior. (Later we can say that the limit is ∞, but for now we will say D.N.E.)



$\lim_{x\to 0} \sin\frac{1}{x}$ does not exist because the function oscillates between two fixed values as *x* approaches 0.

Lesson 1 Exercise

1. Explain why $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

2. Explain why $\lim_{x\to 0}\frac{1}{x^2}$ does not exist.

3. Explain why $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist.

LESSON 1 Solution

1. The right-hand limit and left-hand limit are different. For $x > 2$ we have $\mathbf{|x-2| = x-2}$.

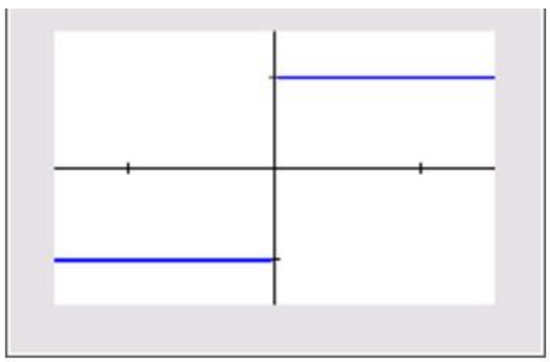
   Therefore, $\lim_{x\to 2^+}\frac{|x-2|}{x-2} = \lim_{x\to 2^+}\frac{x-2}{x-2} = \lim_{x\to 2^+} 1 = 1.$

   Then, for $x < 2$ we have $|x-2| = 2 - x$

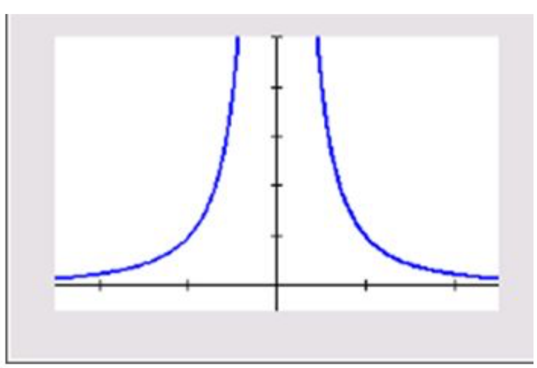
   And $\lim_{x\to 2^-}\frac{|x-2|}{x-2} = \lim_{x\to 2^-}\frac{2-x}{x-2} = \lim_{x\to 2^-}(-1) = -1.$

   Since the limit cannot be both 1 and -1, it does not exist.

   See graph below.



2. As x approaches zero, $1/x^2$ becomes arbitrarily large, so it cannot stay close to any finite number. Therefore $1/x^2$ has no limit as $x \to 0$. See graph below.



3. The graph of sin $(1/x)$ oscillates rapidly between -1and 1 as $x \to 0$, and therefore sin $(1/x)$ has values of both -1 and 1 as it gets close 0. Therefore the limit does not exist since it cannot be both. See graph below.