CALCULUS 1
PREVIEW
WHAT IS CALCULUS

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Simply put, calculus is the study of change and motion. It is quite different from the mathematics that you have studied previously. However, you will need a thorough understanding of the other areas of mathematics that you have studied up to now. This is because a lot of calculus work involves much complex algebra.

Calculus is sometimes also defined as the branch of mathematics that deals with the finding of derivatives and integrals of functions by methods originally based on the summation of infinitesimal differences.

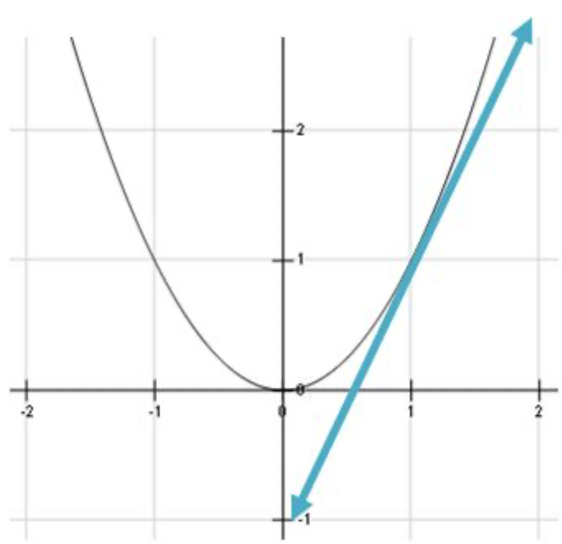
Although many of the ideas of calculus were developed by ancient mathematicians in such diverse places as Greece, China, India, Iraq and Persia, calculus as we know it today was developed in the latter half of the seventeenth century by Gottfried Leibniz (1646 – 1716) in Germany and Isaac Newton (1643 – 1727) in England.

The two major areas of calculus, ***differentiation,*** and ***integration*** deal specifically with two distinct problems—the tangent problem and the area problem. The tangent problem revolves around finding the slope of the tangent line to a parabola at a specific point. While the area problem deals with finding the area of figures whose areas cannot be readily found by the regular methods of arithmetic and geometry.

Later, we will see that the **Fundamental Theorem of Calculus** links differentiation and integration. This connection was discovered by the British mathematician, Isaac Barrow (1630 – 1677). Newton and Leibniz exploited this relationship and used it to develop calculus into a systematic mathematical method.

Tangent to a curve

Let us look at an example of the tangent line problem. The figure below shows an example of the tangent line to a point on a parabola.



The problem involves finding the slope and the equation of the tangent line. In algebra the slope of the line is simple to obtain since it is a constant rate. It is not as easy to find the slope in this case since it is constantly changing. So, we speak of the slope of the tangent line to the curve at a particular point. This is called the ***instantaneous rate of change*** as different from the ***average rate of change*** which was studied in algebra. It is clear, therefore, that in algebra we are interested in finding the slope of a line while in calculus we are interested in finding the slope of a curve.

Example 1.

Find the equation of the tangent line to the curve, $\boldsymbol{y = x^2}$ at the point (**2, 4**).

First find the slope at that point.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{x^2 - 4}{x - 2}$$
$$= \frac{(x-2)(x+2)}{x-2}$$
$$= x + 2 = 2 + 2 = 4$$

The slope = 4.

Next, use the point-slope formula.

Therefore, the tangent line is $\boldsymbol{(y - 4) = 4(x - 2)}$

Example 2.

Find the equation of the tangent line to the curve, $\boldsymbol{y = 3x^3}$ at the point (**1, 3**).

First find the slope at that point.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3x^3 - 3}{x - 1} = \frac{3\left(x^3 - 1\right)}{x - 1}$$
$$= \frac{3(x-1)\left(x^2 + x + 1\right)}{x - 1}$$
$$= 3(1 + 1 + 1) = 9$$

The slope = 9.

Next, use the point-slope formula.

Therefore, the tangent line is $\boldsymbol{(y - 3) = 9(x - 1)}$

Example 3.

Find the equation of the tangent line to the curve, $\boldsymbol{y = x^3 - 3x^2 + 2}$ at the point (**3**, **2**).

First find the slope at that point.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{x^3 - 3x^2 + 2 - 2}{x - 3}$$
$$= \frac{x^3 - 3x^2}{x - 3} = \frac{x^2(x - 3)}{x - 3} = x^2 = 9$$

The slope = 9.

Next, use the point-slope formula.

Therefore, the tangent line is $\boldsymbol{(y - 2) = 9(x - 3)}$

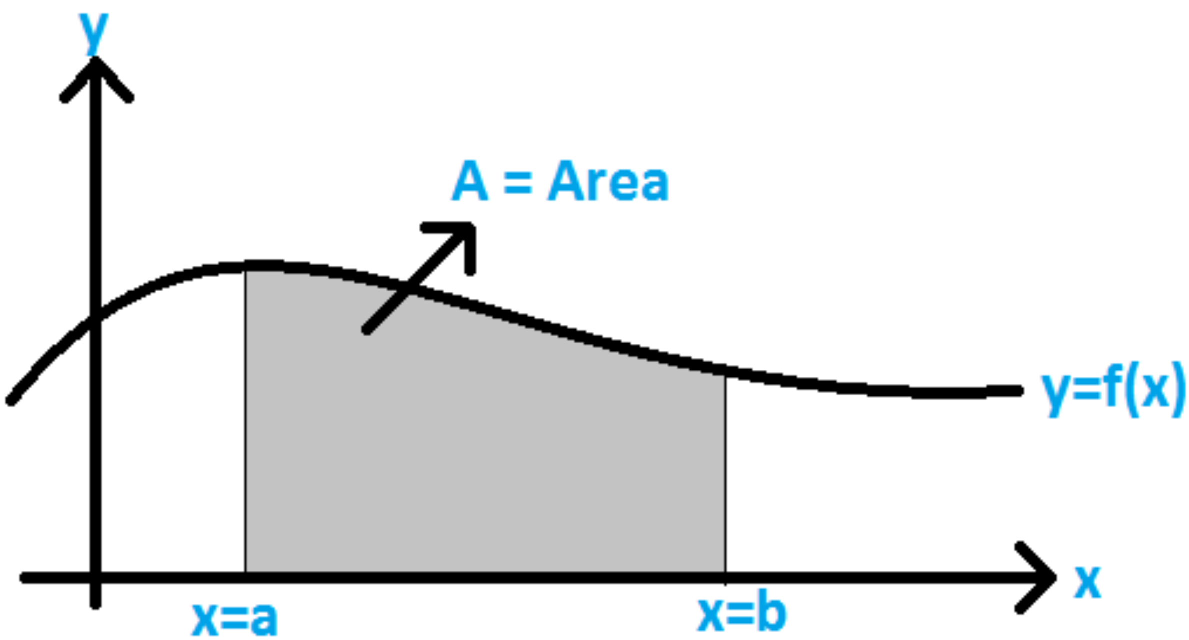
Tangent line exercises

Find the equation of the tangent line.

1. $y = x^2$, at the point (3, 9).
2. $y = x^2$, at the point (0, 0).
3. $y = x^3$, at the point (-1, -1).
4. $y = 3x^2$, at the point (1, 3).

Area under a curve

In the example below we are required to find the area between the curve of $y = f(x)$ and the x – axis, from $x = a$ to $x = b$.



In a later lesson we will examine the concept of the area under the curve in greater detail.

<u>Solutions.</u>

Find the equation of the tangent line.

1. Let $Q = (x, x^2)$ be another point on the graph of the parabola, $y = x^2$. The slope of the line joining (3, 9) and (x, x^2) is $m = \dfrac{x^2-9}{x-3} = \dfrac{(x-3)(x+3)}{x-3} = x + 3, x \neq 3$. As Q approaches (3, 9), x approaches 3. Therefore $m = 6$, and the equation is $y - 9 = 6(x - 3)$, or $y = 6x - 9$.
2. Let $Q = (x, x^2)$ be another point on the graph of the parabola, $y = x^2$. The slope of the line joining (0, 0) and (x, x^2) is $m = \dfrac{x^2-0}{x-0} = x, x \neq 0$. As Q approaches (0, 0), x approaches 0,

and the slope approaches 0. Therefore, the slope of the tangent line is 0, and the equation is y – 0 = 0(x- 0), or y = 0. The tangent is a horizontal line.

3. Let Q = (x, x^3) be another point on the graph of the cubic polynomial, $y = x^3$. The slope of the line joining (-1, -1) and (x, x^3) is.

   $m = \frac{x^3 - (-1)}{x - (-1)} = \frac{x^3 + 1}{x + 1} = \frac{(x+1)(x^2 - x + 1)}{x + 1} = x^2 - x + 1, x \neq -1$. As Q approaches (-1, -1), x approaches -1. And the slope approaches $(-1)^2 - (-1) + 1 = 3$. Therefore, the slope of the tangent is 3, and the equation is y- (-1) = 3(x-(-1), or $y = 3x + 2$.

4. Let Q = (x, $3x^2$) be another point on the graph of the parabola, $y = 3x^2$. The slope of the line joining (1, 3) and (x,$3x^2$) is $m = \frac{3x^2 - 3}{x - 1} = \frac{3(x^2 - 1)}{x - 1} = \frac{3(x-1)(x+1)}{x-1}$

   $$= 3(x+1) = 3(1+1) = 6$$

   As Q approaches (1, 3), x approaches 1. Therefore, the slope of the tangent line is 6, and the equation is $y - 3 = 6(x - 1)$, or $y = 6x - 3$.

Important notes

Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$