LIMITS

WORKED EXAMPLES

Determine the limit by Substitution.

1. $\lim_{x\to 0}\left(x^2-5\right)$
2. $\lim_{x\to 2}\left(x^3+5x^2-7x+1\right)$
3. $\lim_{x\to 0}\dfrac{x^3-6x-8}{x-2}$
4. $\lim_{x\to 8}\dfrac{x^2+64}{x+8}$
5. $\lim_{x\to 5}\sqrt{x^2+14x+49}$
6. $\lim_{x\to 2}\dfrac{x^3-5x^2+2x-4}{x^2-3x+3}$
7. $\lim_{x\to 4}\dfrac{x-\sqrt{x}}{4+\sqrt{x}}$
8. $\lim_{x\to 3}\dfrac{\sqrt{x^2+7}-3}{x+3}$
9. $\lim_{x\to 0}\dfrac{x^2-25}{x^2-4x-5}$
10. $\lim_{x\to 1}\left(x^2-4x\right)^3$

Determine the limit algebraically, if it exists.

1. $\lim_{x\to 2}\dfrac{x^2+x-6}{x-2}$
2. $\lim_{x\to -4}\dfrac{x^2+5x+4}{x^2+3x-4}$
3. $\lim_{x\to 2}\dfrac{x^2-x+6}{x-2}$
4. $\lim_{x\to 4}\dfrac{x^2-4x}{x^2-3x-4}$
5. $\lim_{x\to -3}\dfrac{x^2-9}{2x^2+7x+3}$
6. $\lim_{x\to -1}\dfrac{x^2-4x}{x^2-3x-4}$
7. $\lim_{x\to 4}\dfrac{x^2-x-12}{x-4}$
8. $\lim_{x\to 1}\dfrac{x^4+3x^3-13x^2-27x+36}{x^2+3x-4}$
9. $\lim_{x\to 3}\dfrac{x^2-6x+9}{x^2+2x-3}$
10. $\lim_{x\to 3}\dfrac{x^2-5x+6}{(x-3)^2}$

Limits of Trigonometric Functions

11. $\lim_{x\to 0}\dfrac{\sin 3x}{x}$
12. $\lim_{x\to 0}\dfrac{10\sin x}{6x}$
13. $\lim_{x\to 0}\dfrac{\sin(3x)}{5x}$

14. $\lim_{x\to 0}\dfrac{\sin(2x)}{3x}$

15. $\lim_{x\to 0}\dfrac{\sin(5x)}{\sin(4x)}$

16. $\lim_{x\to 0}\dfrac{2^x+\sin x}{x^4}$

17. $\lim_{x\to 0}\dfrac{1-\cos(3x)}{8x^2}$

18. $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$

19. $\lim_{x\to 0}\dfrac{\tan(3x)}{5x}$

20. $\lim_{x\to 0}\dfrac{\tan(6x)}{8x}$

Limits Involving Infinity

21. $\lim_{x\to\infty}\dfrac{4x+8}{5x}$

22. $\lim_{x\to\infty}\dfrac{5x+5}{7x^2+1}$

23. $\lim_{x\to\infty}\dfrac{5x^2+2}{4x^2+7}$

24. $\lim_{x\to\infty}\dfrac{3x^3+5}{5x^2+1}$

25. $\lim_{x\to\infty}\dfrac{2x^2-4x}{x+1}$

26. $\lim_{x\to\infty}\dfrac{2x^2-4x}{x+1}$

27. $\lim_{x\to\infty}\dfrac{3x^3+2}{5x^2-1}$

28. $\lim_{x\to\infty}\dfrac{3x^2+2}{4x^2-1}$

29. $\lim_{x\to\infty}\dfrac{3-5x}{3x-1}$

30. $\lim_{x\to\infty}\dfrac{6x^2-2x-1}{2x^2+3x+2}$

Limits of Piecewise Functions

Find the indicated limits for the piecewise function:

$$f(x)=\begin{cases} x+1, x<2 \\ x^2-2, 2<x<4 \\ \sqrt{x+5}, x\geq 4 \end{cases}$$

31. $\lim_{x\to -3^-} f(x)$

32. $\lim_{x\to -3^+} f(x)$

33. $\lim_{x\to -3} f(x)$

34. $f(-3)$

35. $\lim_{x\to 2^-} f(x)$

36. $\lim_{x\to 2^+} f(x)$

37. $\lim_{x\to 2} f(x)$

38. $f(2)$

39. $\lim_{x\to 3^-} f(x)$

40. $\lim_{x\to 3^+} f(x)$

41. $\lim_{x\to 3} f(x)$

42. $f(3)$

SOLUTIONS

Note the following:

a. $\lim_{x\to 0}\frac{\sin x}{x}=1$

b. $\lim_{x\to 0}\frac{x}{\sin x}=1$

c. $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ DNE

d. $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

e. (i) If $\lim_{x\to u}\frac{f(x)}{g(x)}=\frac{0}{0}, then \lim_{x\to u}\frac{f(x)}{g(x)}=\lim_{x\to u}\frac{f'(x)}{g'(x)}$

(ii) If $\lim_{x\to u}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}, then \lim_{x\to u}\frac{f(x)}{g(x)}=\lim_{x\to u}\frac{f'(x)}{g'(x)}$

f. $\lim_{x\to\infty}\frac{1}{x}=0$

g. $\lim_{x\to\infty}\frac{x^a}{x^b}$, if a < b. Then, limit = 0.

h. $\lim_{x\to\infty}\frac{Cx^a}{Dx^b}$, if a < b. Then, limit = 0.

i. $\lim_{x\to\infty}\frac{Cx^a}{Dx^b}$, if a = b. Then, limit = $\frac{C}{D}$.

j. $\lim_{x\to\infty}\frac{x^a}{x^b}$, if a > b. Then, limit = ∞.

Determine the limit by substitution.

1. $\lim_{x\to 0}\left(x^2-5\right)=0-5=-5$

2. $\lim_{x\to 2}\left(x^3+5x^2-7x+1\right)=2^3+5\left(2^2\right)-7\left(2\right)+1=15$

3. $\lim_{x\to 0}\dfrac{x^3-6x-8}{x-2}=\dfrac{0-0-8}{0-2}=4$

4. $\lim_{x\to 8}\dfrac{x^2+64}{x+8}=\dfrac{64+64}{8+8}=\dfrac{128}{16}=8$

5. $\lim_{x\to 5}\sqrt{x^2+14x+49}=\sqrt{25+70+49}=12$

6. $\lim_{x\to 2}\dfrac{x^3-5x^2+2x-4}{x^2-3x+3}=\dfrac{2^3-5\left(2^2\right)+2\left(2\right)-4}{2^2-3\left(2\right)+3}=\dfrac{8-20+4-4}{4-6+3}=-12$

7. $\lim_{x\to 4}\dfrac{x-\sqrt{x}}{4+\sqrt{x}}=\dfrac{4-\sqrt{4}}{4+\sqrt{4}}=\dfrac{4-2}{4+2}=\dfrac{2}{6}=\dfrac{1}{3}$

8. $\lim_{x\to 3}\dfrac{\sqrt{x^2+7}-3}{x+3}=\dfrac{\sqrt{3^2+7}-3}{3+3}=\dfrac{1}{6}$

9. $\lim_{x\to 0}\dfrac{x^2-25}{x^2-4x-5}=\dfrac{0-25}{0-0-5}=\dfrac{-25}{-5}=5$

10. $\lim_{x\to 1}\left(x^2-4x\right)^3=\left(1^2-4\right)^3=-3^3=-27$

Determine the limit algebraically if it exists.

1. $\lim_{x\to 2}\dfrac{x^2+x-6}{x-2}=\lim_{x\to -1}\dfrac{\left(x+3\right)\left(x-2\right)}{x-2}=\lim_{x\to -1}x+3=2+3=5$

2. $\lim_{x\to -4}\dfrac{x^2+5x+4}{x^2+3x-4}=\lim_{x\to -4}\dfrac{(x+4)(x+1)}{(x+4)(x-1)}=\lim_{x\to -4}\dfrac{x+1}{x-1}=\dfrac{-4+1}{-4-1}=\dfrac{3}{5}$

3. $\lim_{x\to 2}\dfrac{x^2-x+6}{x-2}$ DNE. Note: $x^2-x+6\to 8,$ while $x-2\to 0.$

4. $\lim_{x\to 4}\dfrac{x^2-4x}{x^2-3x-4}=\lim_{x\to 4}\dfrac{x(x-4)}{(x+1)(x-4)}=\lim_{x\to 4}\dfrac{x}{x+1}=\dfrac{4}{4+1}=\dfrac{4}{5}$

5. $\lim_{x\to -3}\dfrac{x^2-9}{2x^2+7x+3}=\lim_{x\to -3}\dfrac{(x-3)(x+3)}{(2x+1)(x+3)}=\lim_{x\to -3}\dfrac{x-3}{2x+1}=\dfrac{-3-3}{-6+1}=\dfrac{6}{5}$

6. $\lim_{x\to -1}\dfrac{x^2-4x}{x^2-3x-4}=\lim_{x\to -1}\dfrac{x(x-4)}{(x+1)(x-4)}=\lim_{x\to -1}\dfrac{x}{x+1}=\dfrac{-1}{-1+1}=\dfrac{-1}{0}$ DNE

7. $\lim_{x\to 4}\dfrac{x^2-x-12}{x-4}=\lim_{x\to 4}\dfrac{(x-4)(x+3)}{x-4}=\lim_{x\to 4}x+3=4+3=7$

8. $\lim_{x\to 1}\dfrac{x^4+3x^3-13x^2-27x+36}{x^2+3x-4}=\lim_{x\to 1}\dfrac{(x^2+3x-4)(x^2-9)}{x^2+3x-4}$ (divide numerator by the denominator)

$\lim_{x\to 1}x^2-9=1-9=-8$

9. $\lim_{x\to 3}\dfrac{x^2-6x+9}{x^2+2x-3}=\lim_{x\to 3}\dfrac{(x-3)(x-3)}{(x-3)(x+1)}=\lim_{x\to 3}\dfrac{x-3}{x+1}=\dfrac{3-3}{3+1}=\dfrac{0}{4}=0$

10. $\lim_{x\to 3}\dfrac{x^2-5x+6}{(x-3)^2}=\lim_{x\to 3}\dfrac{(x-2)(x-3)}{(x-3)(x-3)}=\lim_{x\to 3}\dfrac{x-2}{x-3}=\dfrac{3-2}{3-3}=\dfrac{1}{0}$ DNE

Limits of Trigonometric Functions

11. $\lim_{x\to 0}\dfrac{\sin 3x}{x}=\lim_{x\to 0}\dfrac{\sin 3x}{x}\times\dfrac{3}{3}=\lim_{x\to 0}\dfrac{\sin 3x}{3x}\times\dfrac{3}{1}=1\times 3=3$

12. $\lim_{x\to 0}\frac{10\sin x}{6x}=\frac{10}{6}\lim_{x\to 0}\frac{\sin x}{x}=\frac{10}{6}\times 1=\frac{5}{6}$

13. $\lim_{x\to 0}\frac{\sin(3x)}{5x}=\lim_{x\to 0}\frac{\sin(3x)}{1}\times\frac{1}{5x}\times\frac{3}{3}=\lim_{x\to 0}\frac{\sin(3x)}{3x}\times\frac{3}{5}=1\times\frac{3}{5}=\frac{3}{5}$

14. $\lim_{x\to 0}\frac{\sin(2x)}{3x}=\lim_{x\to 0}\frac{\sin(2x)}{1}\times\frac{1}{3x}\times\frac{2}{2}=\lim_{x\to 0}\frac{\sin(2x)}{2x}\times\frac{2}{3}=1\times\frac{2}{3}=\frac{2}{3}$

15. $\lim_{x\to 0}\frac{\sin(5x)}{\sin(4x)}=\lim_{x\to 0}\frac{\sin(5x)}{1}\times\frac{1}{\sin(4x)}\times\frac{5x}{5x}\times\frac{4}{4}$

$\lim_{x\to 0}\frac{\sin(5x)}{5x}\times\frac{4x}{\sin(4x)}\times\frac{5}{4}=1\times 1\times\frac{5}{4}$

16. $\lim_{x\to 0}\frac{2^x+\sin x}{x^4}=\lim_{x\to 0}\frac{1}{x^4}\times\lim_{x\to 0}2^x+\sin x=\infty\times 1+0=\infty$

17. $\lim_{x\to 0}\frac{1-\cos(3x)}{8x^2}=\frac{1}{8}\lim_{x\to 0}\frac{1-\cos(3x)}{x^2}=\frac{1}{8}\lim_{x\to 0}\frac{3\sin(3x)}{2x}$ (use L'Hopital's rule twice)

$\frac{1}{8}\lim_{x\to 0}\frac{9\cos\left(3\lim_{x\to 0}x\right)}{2}=\frac{1}{8}\times\frac{9\cos 3(0)}{2}=\frac{1}{8}\times\frac{9\times 1}{2}=\frac{9}{16}$

18. $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}$ (use L'Hopital's rule twice)

$\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$

19. $\lim_{x\to 0}\frac{\tan(3x)}{5x}=\lim_{x\to 0}\frac{\tan(3x)}{1}\times\frac{1}{5x}=\lim_{x\to 0}\frac{\sin(3x)}{\cos(3x)}\times\frac{1}{5x}\times\frac{3}{3}$

$\lim_{x\to 0}\frac{\sin(3x)}{3x}\times\frac{1}{\cos(3x)}\times\frac{3}{5}=1\times 1\times\frac{3}{5}=\frac{3}{5}$

20.
$$\lim_{x\to 0}\frac{\tan(6x)}{8x}=\frac{\tan(6x)}{1}\times\frac{1}{8x}=\lim_{x\to 0}\frac{\sin(6x)}{\cos(6x)}\times\frac{1}{8x}\times\frac{6}{6}$$
$$\lim_{x\to 0}\frac{\sin(6x)}{6x}\times\frac{1}{\cos(6x)}\times\frac{6}{8}=1\times 1\times\frac{6}{8}=\frac{3}{4}$$

Limits Involving Infinity

21. $\lim_{x\to\infty}\frac{4x+8}{5x}=\frac{1}{5}\lim_{x\to\infty}\frac{4x+8}{x}=\frac{4}{5}$ (divide numerator and denominator by *x*)

22. $\lim_{x\to\infty}\frac{5x+5}{7x^2+1}=\lim_{x\to\infty}\frac{\frac{5x}{x^2}+\frac{5}{x^2}}{\frac{7x^2}{x^2}+\frac{1}{x^2}}=\frac{0+0}{7+0}=0$

23. $\lim_{x\to\infty}\frac{5x^2+2}{4x^2+7}=\lim_{x\to\infty}\frac{\frac{5x^2}{x^2}+\frac{2}{x^2}}{\frac{4x^2}{x^2}+\frac{7}{x^2}}=\frac{5+0}{4+0}=\frac{5}{4}$

24. $\lim_{x\to\infty}\frac{3x^3+5}{5x^2+1}=\lim_{x\to\infty}\frac{\frac{3x^3}{x^2}+\frac{5}{x^2}}{\frac{5x^2}{x^2}+\frac{1}{x^2}}=\frac{3x+0}{1+0}=\frac{3(\infty)}{1}=\infty$

25. $\lim_{x\to\infty}\frac{2x^2-4x}{x+1}=\lim_{x\to\infty}\frac{\frac{2x^2}{x}-\frac{4x}{x}}{\frac{x}{x}+\frac{1}{x}}=\frac{2x-4}{1+0}=\infty$

26. $\lim_{x\to\infty}\frac{2x^2-4x}{x+1}=\lim_{x\to\infty}\frac{\frac{2x^2}{x}-\frac{4x}{x}}{\frac{x}{x}+\frac{1}{x}}=\frac{2x-4}{1+0}=2(\infty)=\infty$

27. $\lim_{x\to\infty}\dfrac{3x^3+2}{5x^2-1}=\lim_{x\to\infty}\dfrac{\dfrac{3x^3}{x^2}+\dfrac{2}{x^2}}{\dfrac{5x^2}{x^2}-\dfrac{1}{x^2}}=\dfrac{3x+0}{5-0}=3(\infty)=\infty$

28. $\lim_{x\to\infty}\dfrac{3x^2+2}{4x^2-1}=\lim_{x\to\infty}\dfrac{\dfrac{3x^2}{x^2}+\dfrac{2}{x^2}}{\dfrac{4x^2}{x^2}-\dfrac{1}{x^2}}=\dfrac{3+0}{4-0}=\dfrac{3}{4}$

29. $\lim_{x\to\infty}\dfrac{3-5x}{3x-1}=\lim_{x\to\infty}\dfrac{\dfrac{3}{x}-\dfrac{5x}{x}}{\dfrac{3x}{x}-\dfrac{1}{x}}=\dfrac{0-5}{3-0}=-\dfrac{3}{5}$

30. $\lim_{x\to\infty}\dfrac{6x^2-2x-1}{2x^2+3x+2}=\lim_{x\to\infty}\dfrac{\dfrac{6x^2}{x^2}-\dfrac{2x}{x^2}-\dfrac{1}{x^2}}{\dfrac{2x^2}{x^2}+\dfrac{3x}{x^2}+\dfrac{2}{x^2}}=\dfrac{6-0-0}{2+0+0}=3$

Limits of Piecewise Functions

Find the indicated limits for the piecewise function:

$$f(x)=\begin{cases}x+1, x<2\\ x^2-2, 2<x<4\\ \sqrt{x+5}, x\geq 4\end{cases}$$

31. $\lim_{x\to -3^-} f(x)=\lim_{x\to -3^-} x+1=-3+1=-2$

32. $\lim_{x\to -3^+} f(x)=\lim_{x\to -3^+} x+1=-3+1=-2$

33. $\lim_{x\to -3} f(x)=\lim_{x\to -3} x+1=-3+1=-2$

34. $f(-3)=x+1=-3+1=-2$

35. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x+1 = 2+1 = 3$

36. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 - 2 = 4 - 2 = 2$

37. $\lim_{x\to 2} f(x)\, DNE$

38. $f(2)\, DNE$

39. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 - 2 = 9 - 2 = 7$

40. $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2 - 2 = 9 - 2 = 7$

41. $\lim_{x\to 3} f(x) = \lim_{x\to 3} x^2 - 2 = 9 - 2 = 7$

42. $f(3) = x^2 - 2 = 9 - 2 = 7$