CALCULUS 2
INTEGRATION BY SUBSTITUTION

WORKED EXAMPLES

INTEGRATION BY SUBSTITUTION

Find the integral of the following.

1. $\int (5x+4)^5\,dx$
2. $\int 3t^2(t^3+4)^5\,dx$
3. $\int \sqrt{4x-5}dx$
4. $\int t^2(t^3+4)^{-\frac{1}{2}}\,dt$
5. $\int cos(2x+1)\,dx$
6. $\int sin^{10}\,x\,cos\,d\,x$
7. $\int \frac{sin\,x}{(cos\,x)^5}dx$
8. $\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}dx = \int \frac{\left(x^{\frac{1}{2}}-1\right)^2}{x^{\frac{1}{2}}}\,dx$
9. $\int \sqrt{x^3+x^2}\,(3x^2+2x)dx$
10. $\int (x^2+1)^{15}\,(2x)dx$
11. $\int (x^3-6)^3(3x^2)dx$
12. $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$
13. $\int \frac{2}{\sqrt{3x-7}}dx$
14. $\int x\sqrt{2x+1}\,dx$
15. $\int \sqrt{x}\sqrt{x\sqrt{x}+1}dx =$
    $\int x^{\frac{1}{2}}\sqrt{x^{\frac{3}{2}}+1}dx$
16. $\int x^3\sqrt{x^2-1}dx = \int x^2 * x\sqrt{x^2-1}dx$
    $\int x^3\sqrt{x^2-1}dx = \int x^2 * x\sqrt{x^2+1}dx$
17. $\int (x^2+1)\sqrt{x-2}dx$
18. $\int 3t^3(t^2+4)^5\,dt = \int 3t^2 * t\,(t^2+4)^5$
19. $\int x(x^2-4)^{\frac{7}{2}}$
20. $\int \frac{x}{(1+x^2)^2}dx$
21. $\int \frac{e^t+1}{e^t+t}dt$
22. $\int \frac{e^{\sqrt{y}}}{\sqrt{y}}dy$
23. $\int \frac{e^x}{2+e^x}dx$
24. $\int \frac{x+1}{x^2+2x+19}dx$
25. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$
26. $\int \frac{(t+1)^2}{t^2}dt$
27. $\int \frac{x\,cos(x^2)}{\sqrt{sin(x^2)}}dx$
28. $\int \frac{1+e^x}{\sqrt{x+e^x}}dx$
29. $\int (x+1)\,sin(x^2+2x+3)\,dx$
30. $\int x\,tan(x^2)\,sec(x^2)\,dx$

SOLUTIONS

1. Find $\int (5x+4)^5\,dx$. Let $u = 5x + 4$; $du = 5dx$, $dx = 1/5du$
   Then, $\int (5x+4)^5\,dx = \int u^5 * \frac{1}{5} du = \frac{1}{5} * \frac{1}{6}u^6 + C = \frac{1}{30}(5x+4)^6 + C$

2. Find $\int 3t^2(t^3+4)^5\,dx$. $Let\ u = t^3 + 4;\ du = 3t^2dx;\ dx = \frac{1}{3t^2}du$
   Then, $\int 3t^2(t^3+4)^5\,dx = \int u^5\,du = \frac{u^6}{6} + C = \frac{1}{6}(t^3+4)^6 + C$

3. Find $\int \sqrt{4x-5}dx$ $Let\ u = 4x - 5;\ du = 4dx;\ dx = \frac{1}{4}du$
   Then, $\int \sqrt{4x-5}dx = \frac{1}{4}\int u^{\frac{1}{2}}\,du = \frac{1}{4} * \frac{2}{3}(4x-5)^{\frac{3}{2}} + C = \frac{1}{6}(4x-5)^{\frac{3}{2}}$

4. Find $\int t^2(t^3+4)^{-\frac{1}{2}}\,dt$ $Let\ u = (t^3+4);\ du = 3t^2dx;\ dx = \frac{1}{3t^2}du$
   Then, $\int t^2(t^3+4)^{-\frac{1}{2}}\,dt = \frac{1}{3}\int u^{-\frac{1}{2}}\,du = \frac{1}{3} * 2u^{\frac{1}{2}} + C = \frac{2}{3}(t^3+4)^{\frac{1}{2}} + C$

5. Find $\int cos(2x+1)\,dx$ $Let\ u = 2x + 1;\ du = 2dx;\ dx = \frac{1}{2}du$
   Then, $\int cos(2x+1)\,dx = \frac{1}{2}\int cos\,u\,du = \frac{1}{2}sin\,u + C = \frac{1}{2}sin(2x+1) + C$

6. Find $\int sin^{10}\,x\,cos\,d\,x$ $Let\ u = sin\,x\,;\ du = cos\,x\,dx;\ dx = \frac{1}{cos\,x}du$
   Then, $\int sin^{10}\,x\,cos\,d\,x = \int u^{10}xdu = \frac{1}{11}u^{11} + C$

7. Find $\int \frac{sin\,x}{(cos\,x)^5}dx$ $Let\ u = cos\,x\,;\ du = sin\,x\,dx;\ dx = \frac{1}{sin\,x}du$
   Then, $\int \frac{sin\,x}{(cos\,x)^5}dx = \int \frac{1}{u^5}du = \int u^{-5}\,du = -\frac{1}{4}u^{-4} + C = -\frac{1}{4}(cos\,x)^{-4} + C$

8. Find $\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}dx = \int \frac{\left(x^{\frac{1}{2}}-1\right)^2}{x^{\frac{1}{2}}}\,dx$ $Let\ u = \sqrt{x} - 1;\ \mathrm{du} = \frac{1}{2\sqrt{x}}dx;\ dx = 2\sqrt{x}du$
   Then, $\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}dx = \int \frac{\left(x^{\frac{1}{2}}-1\right)^2}{x^{\frac{1}{2}}}\,dx = 2\int u^2du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(\sqrt{x}-1\right)^3 + C$

9. Find $\int \sqrt{x^3+x^2}\,(3x^2+2x)dx$ $\quad$ *Let* $u = x^3+x^2;\ du = 3x^2+2xdx;\ dx = \frac{1}{3x^2+2x}du$

   Then, $\int \sqrt{x^3+x^2}\,(3x^2+2x)dx = \int u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3+x^2)^{\frac{3}{2}} + C$

10. Find $\int (x^2+1)^{15}\,(2x)dx$ $\quad$ *Let* u $= x^2+1;\ du = 2xdx;\ dx = \frac{1}{2x}du$

    Then, $\int (x^2+1)^{15}\,(2x)dx = \int u^{15}\,du = \frac{1}{16}u^{16} + C = \frac{1}{16}(x^2+1)^{16} + C$

11. Find $\int (x^3-6)^3(3x^2)dx$ $\quad$ *Let* $u\ = x^3-6;\ du = 3x^2dx;\ dx = \frac{1}{3x^2}du$

    Then, $\int (x^3-6)^3(3x^2)dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(x^4-6)^4 + C$

12. Find $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$ $\quad$ *Let* $u = 1+\frac{1}{t};\ du = \frac{1}{-t^2}dt;\ dt = -t^2du$

    Then, $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt = \int u^3 - du = -\frac{1}{4}u^4 + C = -\frac{1}{4}\left(1+\frac{1}{t}\right)^4 + C$

13. Find $\int \frac{2}{\sqrt{3x-7}}dx$ $\quad$ *Let* $u = 3x-7;\ du = 3dx;\ dx = \frac{1}{3}du$

    Then, $\int \frac{2}{\sqrt{3x-7}}dx = \frac{1}{3}\int 2u^{-\frac{1}{2}}du = \frac{2}{3} * 2u^{\frac{1}{2}} + C = \frac{4}{3}(3x-7)^{\frac{1}{2}} + C$

14. Find $\int x\sqrt{2x+1}\,dx$ $\quad$ *Let* $u = 2x+1;\ du = 2dx;\ dx = \frac{1}{2}du;\ x = \frac{u-1}{2}$

    Then, $\int x\sqrt{2x+1}\,dx$

$$= \int x(2x+1)^{\frac{1}{2}}\,dx = \frac{1}{2}\int \frac{1}{2}(u-1)u^{\frac{1}{2}}du = \frac{1}{4}\int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{4} * \frac{2}{5}u^{\frac{5}{2}} - \frac{1}{4} * \frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C$$

15. Find $\int \sqrt{x}\sqrt{x\sqrt{x}+1}dx = \int x^{\frac{1}{2}}\sqrt{x^{\frac{3}{2}}+1}dx$ $\quad$ *Let* $u = x^{\frac{3}{2}}+1;\ du = \frac{3}{2}x^{\frac{1}{2}}dx;\ dx = \frac{2}{3\sqrt{x}}du$

    Then, $\int \sqrt{x}\sqrt{x\sqrt{x}+1}dx = \int x^{\frac{1}{2}}\sqrt{x^{\frac{3}{2}}+1}dx = \frac{2}{3}\int u^{\frac{1}{2}}du = \frac{2}{3} * \frac{2}{3}u^{\frac{3}{2}} + C = \frac{4}{9}\left(x^{\frac{3}{2}}+1\right)^{\frac{3}{2}} + C$

16. Find $\int x^3\sqrt{x^2-1}dx = \int x^2 * x\sqrt{x^2-1}dx$ *Let* $u = x^2-1;\ du = 2xdx;$ $dx = \frac{1}{2x}du;\ x^2 = u+1$

    Then, $\int x^3\sqrt{x^2-1}dx = \int x^2 * x\sqrt{x^2-1}dx = \frac{1}{2}\int(u+1)\left(u^{\frac{1}{2}}\right)du = \frac{1}{2}\int u^{\frac{3}{2}} + u^{\frac{1}{2}}du$

$$= \frac{1}{2} * \frac{2}{5}u^{\frac{5}{2}} + \frac{1}{2} * \frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{5}(x^2-1)^{\frac{5}{2}} + \frac{1}{3}(x^2-1)^{\frac{3}{2}} + C$$

17. Find $\int (x^2+1)\sqrt{x-2}dx$ $\quad$ *Let* $u = x-2;\ du = dx;\ x = u+2, x^2 = (u+2)^2 = u^2+4u+4$

    Then, $\int (x^2+1)\sqrt{x-2}dx = \int (u^2+4u+4+1)\left(u^{\frac{1}{2}}\right)du = \int (u^2+4u+5)\left(u^{\frac{1}{2}}\right)du$

$$= \int u^{\frac{5}{2}} + 4u^{\frac{3}{2}} + 5u^{\frac{1}{2}}du = \frac{2}{7}u^{\frac{7}{2}} + \frac{2}{5} * 4u^{\frac{5}{2}} + \frac{2}{3} * 5u^{\frac{3}{2}} + C$$

$$= \frac{2}{7}(x-2)^{\frac{7}{2}} + \frac{8}{5}(x-2)^{\frac{5}{2}} + \frac{10}{3}(x-2)^{\frac{3}{2}} + C$$

18. Find $\int 3t^3(t^2+4)^5\,dt = \int 3t^2 * t\,(t^2+4)^5$ $\quad Let\ u = t^2+4;\ du = 2tdt, \quad dt = \frac{1}{2t}du; \quad t^2 = u-4$

Then, $\int 3t^3(t^2+4)^5\,dt = \int 3t^2 * t\,(t^2+4)^5 dt = \frac{1}{2}\int 3(u-4)(u)^5\,du =$
$\frac{3}{2}\int u^6 - 4u^5\,du = \frac{3}{2} * \frac{1}{7}u^7 - \frac{3}{2} * \frac{1}{6}4u^6 + C$

$$= \frac{3}{14}(t^2+4)^7 - (t^2+4)^6 + C$$

19. Find $\int x(x^2-4)^{\frac{7}{2}}$ $\quad Let\ u = x^2-4;\ du = 2xdx;\ dx = \frac{1}{2x}du$

Then, $\int x(x^2-4)^{\frac{7}{2}} = \frac{1}{2}\int u^{\frac{7}{2}}\,du = \frac{1}{2} * \frac{2}{9}u^{\frac{9}{2}} + C = \frac{1}{9}(x^2-4)^{\frac{9}{2}} + C$

20. Find $\int \frac{x}{(1+x^2)^2}dx$ $\quad Let\ u = 1+x^2;\ du = 2xdx;\ dx = \frac{1}{2x}du$

Then, $\int \frac{x}{(1+x^2)^2}dx = \frac{1}{2}\int \frac{1}{u^2}du = \frac{1}{2}\int u^{-2}\,du = \frac{1}{2} * -\frac{1}{u} + C = -\frac{1}{2(1+x^2)} + C$

21. Find $\int \frac{e^t+1}{e^t+t}dt$ $\quad Let\ u = e^t + t;\ du = e^t + 1dt;\ dt = \frac{1}{e^t+1}du$

Then, $\int \frac{e^t+1}{e^t+t}dt = \int \frac{1}{u}du = ln|u| + C = ln|e^t + t| + C$ Note: $\left(\int \frac{du}{u} = ln|u| + C\right)$

22. Find $\int \frac{e^{\sqrt{y}}}{\sqrt{y}}dy$ $\quad Let\ u = \sqrt{y};\ du = \frac{1}{2\sqrt{y}}dy;\ dy = 2\sqrt{y}du$

Then, $\int \frac{e^{\sqrt{y}}}{\sqrt{y}}dy = 2\int e^u du = 2e^{\sqrt{y}} + C$

23. Find $\int \frac{e^x}{2+e^x}dx$ $\quad Let\ u = 2 + e^x;\ du = e^x dx;\ dx = \frac{1}{e^x}du$

Then, $\int \frac{e^x}{2+e^x}dx = \int \frac{1}{u}du = ln|2 + e^x| + C$ $\quad$ Note: $\left(\int \frac{du}{u} = ln|u| + C\right)$

24. Find $\int \frac{x+1}{x^2+2x+19}dx$ $\quad Let\ u = x^2 + 2x + 19;\ du = 2x + 2dx;\ dx = \frac{1}{2(x+1)}du$

Then, $\int \frac{x+1}{x^2+2x+19}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}ln|u| + C = \frac{1}{2}ln|x^2 + 2x + 19| + C$

25. Find $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ $\quad Let\ u = e^x + e^{-x};\ du = e^x - e^{-x}dx;\ dx = \frac{1}{e^x - e^{-x}}du$

Then, $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \int \frac{1}{u}du = ln|u| + C = ln|e^x + e^{-x}| + C$

26. Find $\int \frac{(t+1)^2}{t^2}dt$ (This is a case where it is easier not to substitute)

Then, $\int \frac{(t+1)^2}{t^2}dt = \int \frac{t^2+2t+1}{t^2} = \int 1 + \frac{2}{t} + \frac{1}{t^2}dt = t + 2\,ln|t| + \frac{1}{t} + C$

27. Find $\int \frac{x\,cos(x^2)}{\sqrt{sin(x^2)}}dx$ $\quad Let\ u = sin(x^2);\ du = 2x\,cos(x^2)\,dx;\ dx = \frac{1}{2x\,cos(x^2)}du$

Then, $\int \frac{x\,cos(x^2)}{\sqrt{sin(x^2)}}dx = \frac{1}{2}\int u^{-\frac{1}{2}}du = \frac{1}{2}u^{\frac{1}{2}} * 2 + C = \sqrt{sin(x^2)} + C$

28. Find $\int \frac{1+e^x}{\sqrt{x+e^x}}dx$ $\quad Let\ \mathrm{u} = x + e^x;\ du = 1 + e^x dx;\ dx = \frac{1}{1+e^x}du$

Then, $\int \frac{1+e^x}{\sqrt{x+e^x}}dx = \int u^{-\frac{1}{2}}du = 2u^{\frac{1}{2}} + C = 2(x+e^x)^{\frac{1}{2}} + C = 2\sqrt{x+e^x} + C$

29. Find $\int (x+1)\, sin(x^2+2x+3)\, dx$ $\quad Let\ u = x^2+2x+3;\ du = 2x+2dx;\ dx = \frac{1}{2(x+1)}du$

    Then, $\int (x+1)\, sin(x^2+2x+3)\, dx = \frac{1}{2}\int sin\, u\, du = \frac{1}{2}(-cos\, u) + C = -\frac{1}{2}cos(x^2+2x+3) + C$

30. Find $\int x\, tan(x^2)\, sec(x^2)\, dx$ $\quad Let\ u = x^2;\ du = 2xdx;\ dx = \frac{1}{2x}du$

    Then, $\int x\, tan(x^2)\, sec(x^2)\, dx = \frac{1}{2}\int tan\, u\, sec\, u\, du = \frac{1}{2}sec\, u + C = \frac{1}{2}sec(x^2) + C$