CALCULUS 2

INTEGRATION BY PARTS

WORKED EXAMPLES

The formula for integration by parts is: $\int udv = uv - \int vdu$.

The guidelines for selecting **u** and **dv** are:

L: logarithmic functions

I: inverse trig

A: algebraic functions

T: trig functions

E: exponential functions

Note: choose **u** to be the function that comes first.

Example: $\int \sqrt{x} \ln x dx$; let **u** = *lnx* and **dv** = $\sqrt{x}$ dx. This is according to the guideline.

Problems.

Evaluate each indefinite integral using integration by parts.

1. $\int x \cos x dx$
2. $\int x \sin x dx$
3. $\int x^2 \sin x dx$
4. $\int x^2 \cos x dx$
5. $\int x \cos 3x dx$
6. $\int x^2 \cos 3x dx$
7. $\int 3xe^{-x} dx$
8. $\int x \cos 5x dx$
9. $\int x^2 \sin 4x dx$
10. $\int x \cos(5x-1) dx$
11. $\int x^3 \ln x dx$
12. $\int \sin x \ln(\cos x) dx$
13. $\int x^2 e^{-x} dx$
14. $\int 3xe^{-x} dx$
15. $\int \ln(2x+1) dx$
16. $\int x^2 e^{5x} dx$
17. $\int xe^x dx$
18. $\int \cos x \ln(\sin x) dx$
19. $\int e^x \cos x dx$
20. $\int x^2 e^{4x} dx$

SOLUTIONS

1. $\int x\cos x dx$ let $u = x$; $dv = \cos x\, dx$; $du = dx$; $v = \sin x\, dx$.

$= x\sin x - \int \sin x dx \Rightarrow x\sin x - (-\cos x)dx$

$\boxed{x\sin x + \cos x + C}$

2. $\int x\sin x dx$ let $u = x$; $dv = \sin x\, dx$; $du = dx$; $v = -\cos x\, dx$.

$= -x\cos x - \int -\cos x dx \Rightarrow -x\cos x + \int \cos x dx$

$\boxed{-x\cos x + \sin x + C}$

3. $\int x^2 \sin x dx$ let $u = x^2$; $dv = \sin x\, dx$; $du = 2x\, dx$; $v = -\cos x\, dx$.

$= -x^2\cos x - \int -2x\cos x dx \Rightarrow -x^2\cos x + 2\int x\cos x dx$

Now we integrate by parts again using $u = x$; $dv = \cos x\, dx$; $du = dx$; $v = \sin x\, dx$;

Therefore, $\int x^2 \sin x dx = -x^2\cos x + 2\left(x\sin x - \int \sin x\right)dx$

$= x^2\cos x + 2x\sin x - 2(-\cos x) + C$

$\boxed{-x^2\cos x + 2x\sin x + 2\cos x + C}$

4. $\int x^2 \cos x dx$ let $u = x^2$; $dv = \cos x\, dx$; $du = 2x\, dx$; $v = \sin x\, dx$.

$= x^2\sin x - 2\int x\sin x dx \Rightarrow x^2\sin x + 2\int x\sin x dx$

Now we must integrate by parts again using $u = x$; $dv = \sin x\, dx$;

$du = dx$; $v = -\cos x\, dx$

Therefore, $\int x^2\cos x dx = x^2\sin x + 2x\cos x - 2\int \cos x dx$

$\boxed{x^2\sin x + 2x\cos x - 2\sin x + C}$

5. $\int x\cos 3x dx$ let $u = x$; $dv = \cos 3x\, dx$; $du = dx$; $v = 1/3 \sin x\, dx$.

$= \dfrac{x\sin 3x}{3} - \dfrac{1}{3}\int \sin 3x dx \Rightarrow \dfrac{x\sin 3x}{3} - \dfrac{1}{3}\left(-\dfrac{\cos 3x}{3}\right)dx$

$\boxed{\dfrac{1}{3}x\sin 3x + \dfrac{1}{9}\cos 3x + C}$

6. $\int x^2 \cos 3x dx$ let *u = x²; dv = cos3x dx; du = 2x dx; v =1/3 sin x dx.*

$$= \frac{x^2(\sin 3x)}{3} - \int \frac{2x\sin 3x}{3} dx \Rightarrow \frac{x^2(\sin 3x)}{3} - \frac{2}{3}\int x\sin x dx$$

Now we must integrate by parts again using $u = x;\ dv = \sin 3x\ dx;\ du = dx;$

$$v = \frac{\text{-}\cos 3x}{3} dx;$$

$$= \frac{x\sin 3x}{3} - \frac{2}{3}\left(\frac{-x\cos 3x}{3} - \int \frac{\cos 3x}{3}\right) dx$$

$$= \frac{x^2\sin 3x}{3} - \frac{2}{3}\left(\frac{-x\cos 3x}{3} - \frac{1}{3}\left(\frac{\sin 3x}{3}\right)\right) dx$$

$$\boxed{\frac{x^2\sin 3x}{3} + \frac{2x\cos 3x}{9} - \frac{2\sin 3x}{27} + C}$$

7. $\int 3xe^{-x} dx$ let *u = 3x; dv = e⁻ˣ dx; du = 3 dx; v = - e⁻ˣ dx.*

$$= -3e^{-x} - \int -3e^{-x} dx$$

$$\boxed{-3xe^{-x} - 3e^{-x}}$$

8. $\int x\cos 5x dx$ let *u = x; dv = cos5x dx; du = dx; v = 1/5sin x dx.*

$$= \frac{x\sin x}{5} - \int \frac{\sin x}{5} dx \Rightarrow \frac{x\sin x}{5} - \frac{1}{5}\left(\frac{-\cos 5x}{5}\right) + C$$

$$\boxed{\frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C}$$

9. $\int x^2 \sin 4x dx$ let *u = x²; dv = sin4x dx; du =2x dx; v = -1/4cos x dx.*

$$= \frac{-x^2\cos 4x}{4} - \int \frac{-2x\cos 4x}{4} dx \Rightarrow \frac{-x^2\cos 4x}{4} + \frac{1}{2}\int x\cos 4x dx$$

Now we must integrate by parts again, $\int \cos 4x\ dx,\ use\ u = x;$

$$dv = \cos\ 4x\ dx;\ du = dx;\ v = \frac{1}{4}\sin x\ dx$$

$$= \frac{-x^2\cos 4x}{4} + \frac{1}{2}\left(\frac{x\sin 4x}{4} - \frac{1}{4}\int \sin 4x\right) dx$$

$$= \frac{1}{4}x^2\cos 4x + \frac{1}{8}x\sin 4x - \frac{1}{32}(-\cos 4x) + C$$

$$\boxed{\frac{1}{4}x^2\cos 4x + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + C}$$

10. $\int x\cos(5x-1)\,dx$ let $u = x$; $dv = \cos(5x-1)\,dx$; $du = dx$; $v = 1/5\sin x\,dx$.

$$= \frac{x\sin(5x-1)}{5} - \int \frac{\sin(5x-1)}{5}dx \Rightarrow \frac{x\sin(5x-1)}{5} - \frac{1}{5}\left(\frac{-\cos(5x-1)}{5}\right) + C$$

$$\boxed{\frac{1}{5}x\sin(5x-1) + \frac{1}{25}\cos(5x-1) + C}$$

11. $\int x^3 \ln x dx$ let $u = \ln x$; $dv = x^3\,dx$; $du = 1/x\,dx$; $v = 1/4x^4$.

$$= \frac{x^4 \ln x}{4} - \int \frac{x^4}{4} * \frac{1}{x}dx \Rightarrow \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3 dx \Rightarrow \frac{x^4 \ln x}{4} - \frac{1}{4} * \frac{x^4}{4} + C$$

$$\boxed{\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}$$

12. $\int \sin x \ln(\cos x)\,dx$ let $u = \ln(\cos x)$; $dv = \sin x\,dx$; $du = 1/\cos x\ {-\sin x}\,dx = \tan x$;

$v = -\cos x$.

$$= \ln(\cos x)(-\cos x) - \int -\cos x(-\tan x)\,dx \Rightarrow -\cos x(\ln\cos x) - \int \cos x\left(\frac{\sin x}{\cos x}\right)dx$$

$$= -\cos x \ln(\cos x) - \int \sin x dx \Rightarrow -\cos x \ln(\cos x) - (-\cos x) + C$$

$$\boxed{-\cos x \ln(\cos x) + \cos x + C}$$

13. $\int x^2 e^{-x} dx$ let $u = x^2$; $dv = e^{-x}\,dx$; $du = 2x\,dx$; $v = -e^{-x}\,dx$.

$$= x^2 e^{-x} - \int 2x(-e^{-x})\,dx \Rightarrow -x^2 e^{-x} + 2\int xe^{-x}dx$$

Now we must integrate by parts again, $\int xe^{-x}dx$, *use* $u = x$; $dv = e^{-x}\ dx$;

$du = dx$; $v = -e^{-x}\ dx$.

$$= -x^2 e^{-x} + 2\int\left(-xe^{-x} + \int e^{-x}\right)dx$$

$$= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x}dx \Rightarrow -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$\boxed{-e^{-x}(x^2 + 2x + 2) + C}$$

14. $\int 3xe^{-x} dx$ let $u = 3x$; $dv = e^{-x}\,dx$; $du = 3\,dx$; $v = -e^{-x}\,dx$.

$$-3xe^{-x} - \int -3e^{-x}dx \Rightarrow -3xe^{-x} + 3\int e^{-x}dx$$

$$= -3xe^{-x} + 3(-e^{-x}) + C$$

$$\boxed{-3xe^{-x} - 3e^{-x} + C}$$

15. $\int \ln(2x+1)\,dx$ let *u = ln (2x+1); dv = dx; du = 2/2x+1 dx; v = x.*

$$= x\ln(2x+1) - \int \frac{2x}{2x+1}dx \Rightarrow x\ln(2x+1) - \int \frac{(2x+1)-1}{2x+1}dx$$

$$= x\ln(2x+1) - \int 1 - \frac{1}{2x+1}dx$$

$$\boxed{x\ln(2x+1) - x + \frac{1}{2}\ln(2x+1) + C}$$

16. $\int x^2 e^{5x}dx$ let *u = x²; dv = e⁵ˣ dx; du = 2x dx; v = 1/5e⁵ˣ dx.*

$$= \frac{x^2e^{5x}}{5} - \int \frac{2xe^{5x}}{5}dx$$

Now we must integrate by parts again using $u = x;\ dv = e^{5x}dx;\ du = dx;\ v = \frac{1}{5}e^{5x}dx;$

$$Therefore, \int x^2e^{5x}dx = \frac{x^2e^{5x}}{5} - \frac{2}{5}\left(\frac{xe^{5x}}{5} - \frac{1}{5}\int e^{5x}\right)dx$$

$$\boxed{\frac{1}{5}x^2e^{5x} - \frac{2}{25}xe^{5x} + \frac{2}{125}e^{5x} + C}$$

17. $\int xe^x dx$ let *u = x; dv = eˣ dx; du = dx; v = eˣ dx.*

$$= xe^x - \int e^x$$

$$\boxed{xe^x - e^x + C}$$

18. $\int \cos x \ln(\sin x)\,dx$ let *u = ln sin x; du = 1/sin x cos x = cos x/sin x dx; dv = cos x dx;*

v = sin x dx

$$= \ln(\sin x)\sin x - \int \sin x\left(\frac{\cos x}{\sin x}\right)dx \Rightarrow \sin x\ln(\sin x) - \int \cos x dx$$

$$= \sin x \ln(\sin x) - \sin x + C$$

$$\boxed{\sin x(\ln \sin x - 1) + C}$$

19. $\int e^x \cos x dx$ *let u = cos x; dv = eˣ dx; du = -sin x dx; v = eˣ dx.*

$= e^x \cos x + \int e^x \sin x dx$

Now we must integrate by parts again using $u = \sin x$; $dv = e^x dx$; $du = -\cos x \, dx$; $v = e^x$

$\Rightarrow e^x \cos x + e^x \sin x - \int e^x \cos x dx$

Here we arrive at the original function again, so we add $\int e^x$cosx dx to both sides.

$$\begin{aligned} \textit{Therefore}, \int e^x \cos x dx &= e^x \cos x + e^x \sin x - \int e^x \cos x dx \\ + \int e^x \cos x dx &= \qquad\qquad\qquad\qquad + \int e^x \cos x dx \\ \hline 2\int e^x \cos x dx &= e^x \cos x + e^x \sin x + C \quad (\text{Next divide by } 2) \end{aligned}$$

$$\boxed{\frac{e^x (\cos x + \sin x)}{2} + C}$$

20. $\int x^2 e^{4x} dx$ let $u = x^2$; $dv = e^{4x}$ dx; $du = 2x\ dx$; $v = 1/4e^{4x}\ dx$.

$$= \frac{x^2 e^{4x}}{4} - \int \frac{2xe^{4x}}{4} dx \Rightarrow \frac{x^2 e^{4x}}{4} - \frac{1}{2} \int \frac{xe^{4x}}{4} dx \Rightarrow$$

$$= \frac{x^2 e^{4x}}{4} - \frac{1}{2}\left(\frac{xe^{4x}}{4} - \int \frac{e^{4x}}{4} \right) dx \Rightarrow \frac{x^2 e^{4x}}{4} - \frac{1}{2}\left(\frac{xe^{4x}}{4} - \frac{1}{4} * \frac{e^{4x}}{4} \right) + C$$

$$\boxed{\frac{1}{4} x^2 e^{4x} - \frac{1}{8} xe^{4x} + \frac{1}{32} e^{4x} + C}$$