LESSON 2

Sum and Difference Identities

Angle sum identities and angle difference identities can be used to find the function values of any angles however, the most practical use is to find exact values of an angle that can be written as a sum or difference using the familiar values for the sine, cosine, and tangent of the 30°, 45°, 60° and 90° angles and their multiples. Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. The Sum and Difference Identities are shown below.

Sum and Difference Identities:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

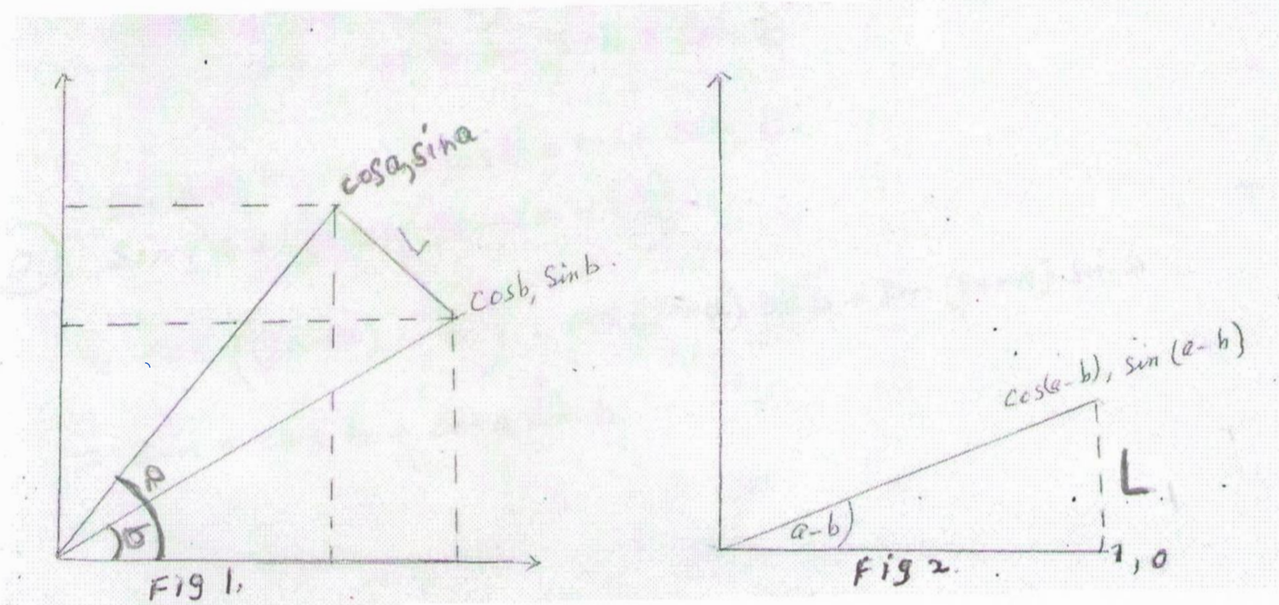
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

Angle sum identities and angle difference identities can be used to find the function values of any angles however, the most practical use is to find exact values of an angle that can be written as a sum or difference using the familiar values for the sine, cosine, and tangent of the 30°, 45°, 60° and 90° angles and their multiples.

Proof of (cos a – cos b) = cos a cos b + sin a sin b

Proof of (cos a – cos b) = cos a cos b + sin a sin b



In fig 1,

$L^2 = (\cos a - \cos b)^2 + (\sin a - \sin b)^2$

$= \cos^2 a - 2\cos a \cos b + \cos^2 b + \sin^2 a - 2\sin a \sin b + \sin^2 b$ (Note: $\sin^2 a + \cos^2 a = 1$)

$= 1 + 1 - 2\cos a \cos b - 2\sin a \sin b$

$L^2 = 2 - 2\cos a \cos b - 2\sin a \sin b$ (equation 1).

In fig 2,

$L^2 = \{\cos (a - b) - 1\}^2 + \{\sin (a - b) - 0\}^2$

$= \cos^2 (a - b) - 2\cos (a - b) + 1 + \sin^2 (a - b)$

$= 1 + 1 - 2\cos (a - b)$

$L^2 = 2 - 2 \cos (a - b)$ (equation 2).

Set fig 2 = fig 1

$2 - 2 \cos (a - b) = 2 - 2 \cos a \cos b - 2 \sin a \sin b$ subtract 2

$- 2 \cos (a - b) = - 2 \cos a \cos b - 2 \sin a \sin b$ factor by – 2

$- 2 \cos (a - b) = - 2 (\cos a \cos b + \sin a \sin b)$ add 2

$\cos (a - b) = \cos a \cos b + \sin a \sin b$

How to use the Sum and Difference Identities

Example 1
Find the exact value of cos 195°.
cos (a + b) = cos a cos b – sin a sin b

$$\cos 195^{o} = \cos\left(150^{o} + 45^{o}\right) = \frac{-\sqrt{3}}{2} * \frac{\sqrt{2}}{2} - \frac{1}{2} * \frac{\sqrt{2}}{2} = \frac{-\sqrt{6}}{2} - \frac{\sqrt{3}}{2}$$

$$= \frac{-\sqrt{6} - \sqrt{3}}{4}$$

Example 2
Find the exact value of sin 195°.
sin (a + b) = sin a cos b +cos a sin b

$$\sin 195^{o} = \sin(150 + 45) = \frac{1}{2} * \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} * \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$

$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

Example 3
Find the exact value of sin 165°.
sin(a + b) = sin a cos b + cos a cos b

$$\sin 165 = \sin(120 + 45) = \frac{\sqrt{3}}{2} * \frac{\sqrt{2}}{2} - \frac{1}{2} * \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example 4
Find the exact value of cos 105°.
cos(a + b) = cos a cos b – sin a sin b

$$\cos 105° = \cos(60 + 45) = \frac{1}{2} * \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} * \frac{\sqrt{2}}{2} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

Example 5
Find the exact value of cos –15°.
cos(a - b) = cos a cos b + sin a sin b

$$\cos -15° = \cos(30 - 45) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Lesson 2 Exercise

Use the angle sum identity to find the exact value of each.

1) $\cos 105^\circ$

2) $\sin 195^\circ$

3) $\cos 195^\circ$

4) $\cos 165^\circ$

5) $\cos 285^\circ$

6) $\cos 255^\circ$

7) $\sin 105^\circ$

8) $\sin 285^\circ$

9) $\cos 75^\circ$

10) $\sin 255^\circ$

SOLUTIONS

Lesson 1 Exercise

1. sin θ + cos θ =1

tan θ = $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \dfrac{\sqrt{3}}{2} * \dfrac{2}{1} = \sqrt{3}$

$$\sin^2\theta + \left(\frac{1}{2}\right)^2 = 1$$

$$\sin^2\theta + \frac{1}{4} = 1$$

$$\sin^2\theta = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

2. sin θ + cos θ =1

tan θ = $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{\sqrt{7}}{4}}{\frac{3}{4}} = \dfrac{\sqrt{7}}{4} * \dfrac{4}{3} = \dfrac{\sqrt{7}}{3}$

$$\sin^2\theta + \left(\frac{3}{4}\right)^2 = 1$$

$$\sin^2\theta + \frac{9}{16} = 1$$

$$\sin^2\theta = 1 - \frac{9}{16} = \frac{7}{16}$$

$$\sin\theta = \frac{\sqrt{7}}{4}$$

3. sin θ + cos θ =1

tan θ = $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{3}{5}}{\frac{4}{5}} = \dfrac{3}{5} * \dfrac{5}{4} = \dfrac{3}{4}$

$$\left(\frac{3}{5}\right)^2 + \cos\theta = 1$$

$$\frac{9}{25} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\cos\theta = \frac{4}{5}$$

4. sin θ + cos θ =1

tan θ = $\frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{4}}{\frac{\sqrt{15}}{4}} = \frac{1}{4} * \frac{4}{\sqrt{15}} \Rightarrow \frac{1}{\sqrt{15}} * \frac{\sqrt{15}}{\sqrt{15}} = \frac{\sqrt{15}}{15}$

$$\left(\frac{1}{4}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{16} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{1}{16} = \frac{15}{16}$$

$$\cos\theta = \frac{\sqrt{15}}{4}$$

5. sin θ + cos θ =1

tan θ = $\frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{8}}{\frac{\sqrt{55}}{8}} = \frac{3}{8} * \frac{8}{\sqrt{55}} \Rightarrow \frac{3}{\sqrt{55}} * \frac{\sqrt{55}}{\sqrt{55}} = \frac{3\sqrt{55}}{55}$

$$\left(\frac{3}{8}\right)^2 + \cos^2\theta = 1$$

$$\frac{9}{64} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{9}{64} = \frac{55}{64}$$

$$\cos\theta = \frac{\sqrt{55}}{8}$$

6. sin θ + cos θ =1

tan θ = $\frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{4}}{\frac{\sqrt{7}}{4}} = \frac{3}{4} * \frac{4}{\sqrt{7}} \Rightarrow \frac{3}{\sqrt{7}} * \frac{\sqrt{7}}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$

$$\left(\frac{3}{4}\right)^2 + \cos^2\theta = 1$$

$$\frac{9}{16} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{9}{16} = \frac{7}{16}$$

$$\cos\theta = \frac{\sqrt{7}}{4}$$

7. sin θ + cos θ =1 **tan θ =** $\frac{\sin\theta}{\cos\theta}=\frac{\frac{\sqrt{15}}{4}}{\frac{1}{4}}=\frac{\sqrt{15}}{4}*\frac{4}{1}=\sqrt{15}$

$$\sin^2\theta+\left(\frac{1}{4}\right)^2=1$$
$$\sin^2\theta+\frac{1}{16}=1$$
$$\sin^2\theta=1-\frac{1}{16}=\frac{15}{16}$$
$$\sin\theta=\frac{\sqrt{15}}{4}$$

8. sin θ + cos θ =1 **tan θ =** $\frac{\sin\theta}{\cos\theta}=\frac{\frac{4}{5}}{\frac{3}{5}}=\frac{4}{5}*\frac{5}{3}=\frac{4}{3}$

$$\sin^2\theta+\left(\frac{3}{5}\right)^2=1$$
$$\sin^2\theta+\frac{9}{25}=1$$
$$\sin^2\theta=1-\frac{9}{25}=\frac{16}{25}$$
$$\sin\theta=\frac{4}{5}$$

9. sin θ + cos θ =1 **tan θ =** $\frac{\sin\theta}{\cos\theta}=\frac{\frac{5}{8}}{\frac{\sqrt{39}}{8}}=\frac{5}{8}*\frac{8}{\sqrt{39}}\Rightarrow\frac{5}{\sqrt{39}}*\frac{\sqrt{39}}{\sqrt{39}}=\frac{5\sqrt{39}}{39}$

$$\left(\frac{5}{8}\right)^2+\cos^2\theta=1$$
$$\frac{25}{64}+\cos^2\theta=1$$
$$\cos^2\theta=1-\frac{25}{64}=\frac{39}{64}$$
$$\cos\theta=\frac{\sqrt{39}}{8}$$

10. sin θ + cos θ =1 **tan θ =** $\frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{39}}{8}}{\frac{5}{8}} = \frac{\sqrt{39}}{8} * \frac{8}{5} = \frac{\sqrt{39}}{5}$

$$\sin^2\theta + \left(\frac{5}{8}\right)^2 = 1$$

$$\sin^2\theta + \frac{25}{64} = 1$$

$$\sin^2\theta = 1 - \frac{25}{64} = \frac{39}{64}$$

$$\sin\theta = \frac{\sqrt{39}}{8}$$

Lesson 2 Exercise

1. $\frac{\sqrt{2}-\sqrt{6}}{4}$
2. $\frac{\sqrt{2}-\sqrt{6}}{4}$
3. $\frac{-\sqrt{6}-\sqrt{2}}{4}$
4. $\frac{-\sqrt{6}-\sqrt{2}}{4}$
5. $\frac{\sqrt{6}-\sqrt{2}}{4}$
6. $\frac{\sqrt{2}-\sqrt{6}}{4}$
7. $\frac{\sqrt{6}+\sqrt{2}}{4}$
8. $\frac{-\sqrt{6}-\sqrt{2}}{4}$
9. $\frac{\sqrt{2}-\sqrt{6}}{4}$
10. $\frac{-\sqrt{6}-\sqrt{2}}{4}$