LESSON 3

SOLVING BY GAUSS JORDAN ELIMINATION

A **matrix** can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an **augmented matrix**.

Steps to write a matrix in augmented form from a given system of linear equations.

1. Write the coefficients of the *x*-terms as the numbers down the first column.
2. Write the coefficients of the *y*-terms as the numbers down the second column.
3. If there are *z*-terms, write the coefficients as the numbers down the third column.
4. Draw a vertical line and write the constants to the right of the line.

Shown below is a system of linear equations and the augmented matrix obtained from it.

$$\begin{aligned} 2x + y - 2z &= -9 \\ x + 3y - z &= -17 \\ 3x + 4y - z &= -20 \end{aligned} \qquad \left|\begin{array}{ccc|c} 2 & 1 & -2 & -9 \\ 1 & 3 & -1 & -17 \\ 3 & 4 & -1 & -20 \end{array}\right|$$

The goal when solving a system of equations is to place the augmented matrix into reduced row-echelon form, (RREF) if possible. There are three elementary row operations that you may use to accomplish placing a matrix into reduced row-echelon form. Each of these can be satisfied using the elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Reduced Row Echelon Form

1. **You can multiply any row by a constant (other than zero).**
2. **You can switch any two rows.**
3. **You can add two rows together.**

The goals for solving reduced row echelon elimination are as follows.

1. Get a 1 in the upper-left corner.
2. Get 0s underneath the 1 in the first column.
3. In the third row, get a 0 under the 1.
4. Get a 1 in the second row, second column.
5. Get a 0 under the 1 you created in row two.
6. Get another 1, this time in the third row, third column.
7. Get a 0 in row two, column three.
8. Get a 0 in row one, column three.
9. Get a 0 in row one, column two.

This matrix, now in reduced row echelon form, is the solution to the system.

Example 1. **NOTE.** $-R_2 + R_1 \Rightarrow R_1$ means negative Row 2 plus Row 1 is the new Row 1.

$$\begin{aligned} 2x + y - 2z &= -9 \\ x + 3y - z &= -17 \\ 3x + 4y - z &= -20 \end{aligned} \Rightarrow \left|\begin{array}{ccc|c} 2 & 1 & -2 & -9 \\ 1 & 3 & -1 & -17 \\ 3 & 4 & -1 & -20 \end{array}\right| \; -R_2 + R_1 \Rightarrow R_1 \Rightarrow \left|\begin{array}{ccc|c} 1 & -2 & -1 & 8 \\ 1 & 3 & -1 & -17 \\ 3 & 4 & -1 & -20 \end{array}\right|$$

$$-R_1 + R_2 \Rightarrow R_2 \Rightarrow \left|\begin{array}{ccc|c} 1 & -2 & -1 & 8 \\ 0 & 5 & 0 & -25 \\ 3 & 4 & -1 & -20 \end{array}\right| \; -3R_1 + R_3 \Rightarrow R_3 \Rightarrow \left|\begin{array}{ccc|c} 1 & -2 & -1 & 8 \\ 0 & 5 & 0 & -25 \\ 0 & 10 & 2 & -44 \end{array}\right|$$

$$-2R_2 + R_3 \Rightarrow R_3 \Rightarrow \left|\begin{array}{ccc|c} 1 & -2 & -1 & 8 \\ 0 & 5 & 0 & -25 \\ 0 & 0 & 2 & 6 \end{array}\right| \; 1/2R_3 \Rightarrow R_3 \Rightarrow \left|\begin{array}{ccc|c} 1 & -2 & -1 & 8 \\ 0 & 5 & 0 & -25 \\ 0 & 0 & 1 & 3 \end{array}\right|$$

$$1/5R_2 \Rightarrow R_2 \Rightarrow \left|\begin{array}{ccc|c} 1 & -2 & -1 & 8 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right| \; R_3 + R_1 \Rightarrow R_1 \Rightarrow \left|\begin{array}{ccc|c} 1 & -2 & 0 & 11 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right|$$

$$2R_2 + R_1 \Rightarrow R_1 \Rightarrow \left|\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right|$$

Example 2.

$$\begin{aligned} 2b + c + z &= 11 \\ 3b + 4c + z &= 19 \\ 3b + 6c + 5z &= 43 \end{aligned} \Rightarrow \left|\begin{array}{ccc|c} 2 & 1 & 1 & 11 \\ 3 & 4 & 1 & 19 \\ 3 & 6 & 5 & 43 \end{array}\right| \; R_2 - R_1 \Rightarrow R_1 \Rightarrow \left|\begin{array}{ccc|c} 1 & 3 & 0 & 8 \\ 3 & 4 & 1 & 19 \\ 3 & 6 & 5 & 43 \end{array}\right|$$

$$\Rightarrow -R_3 + R_2 \Rightarrow R_2 \left|\begin{array}{ccc|c} 1 & 3 & 0 & 8 \\ 0 & -2 & -4 & -24 \\ 3 & 6 & 5 & 43 \end{array}\right| \; -3R_1 + R_3 \Rightarrow R_3 \Rightarrow \left|\begin{array}{ccc|c} 1 & 3 & 0 & 8 \\ 0 & -2 & -4 & -24 \\ 0 & -3 & 5 & 19 \end{array}\right|$$

$$\Rightarrow -3/2R_2 + R_3 \Rightarrow R_3 \left|\begin{array}{ccc|c} 1 & 3 & 0 & 8 \\ 0 & -2 & -4 & -24 \\ 0 & 0 & 11 & 55 \end{array}\right| \Rightarrow -1/11R_3 \Rightarrow R_3 \left|\begin{array}{ccc|c} 1 & 3 & 0 & 8 \\ 0 & -2 & -4 & -24 \\ 0 & 0 & 1 & 5 \end{array}\right|$$

$$\Rightarrow -1/2R_2 \Rightarrow R_2 \left|\begin{array}{ccc|c} 1 & 3 & 0 & 8 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 1 & 5 \end{array}\right| \Rightarrow -2R_3 + R_2 \Rightarrow R_2 \left|\begin{array}{ccc|c} 1 & 3 & 0 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right|$$

$$-3R_2 + R_1 \Rightarrow R_1 \left|\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right|$$

Lesson 3 Exercise

Solve by using Gauss Jordon Elimination.

1. $x + 3y - 2z = -11$
   $-2x - 5y + 3z = 17$
   $4x - z = 1$

2. $-2x - 4y - 5z = 11$
   $-x + 4z = -25$
   $-3x - 5y + z = -25$

3. $4x - 2y = 2$
   $5x - 2y + z = 7$
   $3x + 4y - z = 3$

4. $x - y + 2z = -1$
   $-3x + 3y + 5z = 3$
   $2x - 2y = -2$

5. $2x + 5y + z = -12$
   $-x + 4y + 3z = -4$
   $5x - 2z = -13$

6. $2x + 8y + 4z = 2$
   $2x + 5y + z = 5$
   $4x + 10y - z = 1$

SOLUTIONS

Lesson 1 Exercise

1. (- 2, - 3, 3) 2. (- 4, 0 – 2) 3. (- 1, - 4, - 4,) 4. (4, 2, 0)

5. (1, 3, 1) 6. (0, 0, - 5) 7. (- 5, - 5, 3) 8. No Unique Solution.

Lesson 2 Exercise

1. (- 2, - 4, 5) 2. (1, - 4, - 5) 3. (- 2, 0, - 5) 4. (0, -1, -1)

5. (0, 3, 2) 6. (2, 2, 4) 7.(0, 3, 5) 8. (- 5, 4, -3)

Lesson 3 Exercise

1. (1, - 2, 3) 2. (5, 1, - 5) 3. (1, 1, 4) 4. (-1, 0, 0)

5. (-3, - 1, -1) 6. (11, - 4, 3)

Further Practice.

Use any method to solve the following systems of equations.

1. $\begin{aligned} x+2y-z&=9 \\ 2x-z&=-2 \\ 3x+5y+2z&=22 \end{aligned}$

2. $\begin{aligned} x+y+z&=6 \\ 2x-y+z&=3 \\ 3x-z&=0 \end{aligned}$

3. $\begin{aligned} x+y+z&=2 \\ -x+3y+2z&=8 \\ 4x+y&=4 \end{aligned}$

4. $\begin{aligned} 4x+y-3z&=11 \\ 2x-3y+2z&=9 \\ x+y+z&=-3 \end{aligned}$

5. $\begin{aligned} 2x+2z&=2 \\ 5x+3y&=4 \\ 3y-4z&=4 \end{aligned}$

6. $\begin{aligned} 6y+4z&=-12 \\ 3x+3y&=9 \\ 2x-3z&=10 \end{aligned}$

7. $\begin{aligned} 2x+4y+z&=-4 \\ 2x-4y+6z&=13 \\ 4x-2y+z&=6 \end{aligned}$

8. $\begin{aligned} 3x-2y+4z&=1 \\ x+y-2z&=3 \\ 2x-3y+6z&=8 \end{aligned}$

9. $\begin{aligned} 5x-3y+2z&=3 \\ 2x+4y-z&=7 \\ x-11y+4z&=3 \end{aligned}$

10. $\begin{aligned} 3x+3y+5z&=1 \\ 3x+5y+9z&=0 \\ 5x+9y+17z&=0 \end{aligned}$

11. $\begin{aligned} 2x+y+3z&=1 \\ 2x+6y+8z&=3 \\ 6x+8y+18z&=5 \end{aligned}$

12. $\begin{aligned} x+2y-7z&=-4 \\ 2x+y+z&=13 \\ 3x+9y-36z&==-33 \end{aligned}$

13. $\begin{aligned} 2x+y-3z&=4 \\ 4x+2z&=10 \\ -2x+3y-13z&=-8 \end{aligned}$

14. $\begin{aligned} 4x-y+5z&=11 \\ x+2y-z&=5 \\ 5x-8y+13z&=7 \end{aligned}$

15. $\begin{aligned} x-2y+5z&=2 \\ 3x+2y-z&=-2 \end{aligned}$

Solutions

1. $x=-1,\ y=5,\ z=0$
2. $x=1,\ y=2,\ z=3$
3. $x=0,\ y=4,\ z=-2$
4. $x=2,\ y=-3,\ z=-2$
5. $x=-4,\ y=8,\ z=5$
6. $x=5,\ y=-2,\ z=0$
7. $x=1/2,\ y=-3/2,\ z=1$
8. No solution
9. No solution
10. $x=1,\ y=-3/2,\ z=1/2$
11. $x=3/10,\ y=2/5,\ z=0$
12. Infinite solutions
13. $x=5/3,\ y=17/3,\ z=5/3$
14. Infinite solutions
15. $x=-z,\ y=2z-1$