PARTIAL FRACTION DECOMPOSITION

Example

$$\frac{-x+5}{x^2+2x-3}=\frac{A}{x-1}+\frac{B}{x+3}$$

$$-x+5=A(x+3)+B(x-1)$$

$$-x+5=Ax+3A+Bx-B$$

$$-x+5=(A+B)x+(3A-B)$$

In elementary school you would have learned how to add or subtract fractions. For example, when you subtract the second fraction from the first in the problem below, you proceed as follows

$\frac{5}{8}-\frac{1}{4}=\frac{5-2}{8}=\frac{3}{8}$ First multiply the numerators by the least common denominator.

The procedure is the same when working with algebraic fractions.

For example,

$$\frac{1}{x-1}-\frac{2}{x+3}$$

$$\frac{x+3-2(x-1)}{(x-1)(x+3)}$$

Then simplify and solve.

$$\frac{x+3-2x+2}{(x-1)(x+3)}=\frac{-x+5}{(x-1)(x+3)}$$

$$\frac{-x+5}{x^2+2x-3}$$

The reverse process of adding or subtracting fractions, that is, the splitting up of a rational expression into partial fractions is what is termed ***partial fraction decomposition***.

Note that partial fraction decomposition depends on a few important rules.

1. For every linear factor, $(x - a)$, in the denominator there exists a partial fraction in the form $\mathbf{A}/(x - a)$.
2. For every repeated quadratic factor such as $(x - a)^2$ in the denominator, there exists two partial fractions of the form $\mathbf{A}/(x - a)$ and $\mathbf{B}/(x - a)^2$. Similarly, for factors such as $(x - a)^3$ there will be three partial fractions, $\mathbf{A}/(x-a)$, $\mathbf{B}/(x-a)^2$ and $\mathbf{C}/(x-a)^3$.
3. For every irreducible quadratic factor such as $(x^2 + ax + b)$ there exists a partial fraction $(\mathbf{C}x + \mathbf{D})/(x^2 + ax + b)$.
4. If the degree of the numerator is equal to, or greater than, that of the denominator, divide the numerator by the denominator until a remainder that is lower in degree than the denominator is obtained. Now follow rules 1, 2 and 3 to decompose the remaining rational expression.

SECTION 1

LINEAR FACTORS

Examples in the use of rule (1).

Example 1.

Find the partial fraction decomposition of

$$\frac{-x+5}{x^2+2x-3}$$

Step 1: First factor the denominator, then set up the partial fraction decomposition with unknown constants ***A*** and ***B*** such that a constant is the numerator of each of the linear factors,

$$\frac{-x+5}{x^2+2x-3}=\frac{A}{x-1}+\frac{B}{x+3}$$

Step 2: The objective is to solve for ***A*** and ***B***. Therefore, to do this we must multiply both sides of the equation by the least common denominator,

$$(x^2+2x-3)\left(\frac{-x+5}{x^2+2x-3}\right)=(x^2+2x-3)\left(\frac{A}{x-1}\right)+(x^2+2x-3)\left(\frac{B}{x+3}\right)$$

Doing this we obtain,

$$-x+5=A(x+3)+B(x-1)$$

Step 3: We now need to find values for ***A*** and ***B*** that make both sides equal. To do this we apply the distributive property to the right side of the equation. We now have,

$$-x+5=Ax+3A+Bx-B$$

Step 4: Now we rearrange the right side. Keep the ***x***-terms together and the constants together.

$$-x+5=Ax+Bx+3A-B$$

Factor out the *x* on the right side.

$$-x+5=(A+B)x+(3A-B)$$

Step 5: Since the two polynomials are equal it follows that the coefficients of like powers of ***x*** are also equal. Therefore, ***A*** and ***B*** satisfy the two following equations. By solving this system of equations in two unknowns we obtain the values for ***A*** and ***B***.

$A+B=-1$ By addition

$3A-B=5$

$$4A = 4$$

$$A = 1$$ By Substitution

$$B = -2$$

Therefore, the partial fraction decomposition is, $\frac{-x+5}{x^2+2x-3} = \frac{1}{x-1} - \frac{2}{x+3}$

<u>Example 2</u>

Find the partial fraction decomposition of

$$\frac{5x-4}{x^2-x-2}$$

Step 1: First factor the denominator, then set up the partial fraction decomposition with unknown constants ***A*** and ***B*** such that a constant is the numerator of each of the linear factors,

$$\frac{5x-4}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}$$

Step 2: The objective is to solve for ***A*** and ***B***. Therefore, to this we must multiply both sides of the equation by the least common denominator,

$$(x^2-x-2)\left(\frac{5x-4}{x^2-x-2}\right) = (x^2-x-2)\left(\frac{A}{x-2}\right) + (x^2-x-2)\left(\frac{B}{x+1}\right)$$

Doing this we obtain,

$$5x-4 = A(x+1) + B(x-2)$$

Step 3: We now need to find values for ***A*** and ***B*** that make both sides equal. To do this we apply the distributive property to the right side of the equation. We now have,

$$5x-4 = Ax + A + Bx - 2B$$

Step 4: Now we rearrange the right side. Keep the ***x***-terms together and the constants together.

$$5x-4=Ax+Bx+A-2B$$

Factor out the *x* on the right side.

$$5x-4=(A+B)x+(A-2B)$$

Step 5: Since the two polynomials are equal it follows that the coefficients of like powers of *x* are also equal. Therefore, ***A*** and ***B*** satisfy the two following equations. By solving this system of equations in two unknowns we obtain the values for ***A*** and ***B***.

A + B = 5

<u>***A - 2B = -4***</u> by subtraction

3B= 9

B= 3

A= 2 substituting ***B = 3*** in the first equation.

Therefore, the partial fraction decomposition is, $\frac{5x-4}{x^2-x-2}=\frac{2}{x-2}+\frac{3}{x+1}$

Independent Practice 1

Find the partial fraction decomposition.

1. $\frac{7x-1}{x^2+x-2}$
2. $\frac{4x+6}{x^2-9}$
3. $\frac{2x+6}{x^2+3x+2}$

4. $\dfrac{4x-3}{2x^2-3x+1}$

5. $\dfrac{(-2)x+19}{6x^2+7x-3}$

6. $\dfrac{-x-8}{2x^2-3x-2}$

Solutions to independent practice 1.

1. $\dfrac{7x-1}{x^2+x-2}=\dfrac{A}{x-1}+\dfrac{B}{x+2}$

$7x-1=A(x+2)+B(x-1)$ (multiply both sides by the common denominator)

$7x-1=Ax+2A+Bx-B$ (apply the distributive property)

$7x-1=Ax+Bx+2A-B$ (arrange like terms in descending powers of *x*)

$7x-1=(A+B)x+(2A-B)$ (factor out powers of *x*)

A+B = 7

<u>2A-B = -1</u> (set coefficients equal)

3A = 6

A = 2 (by addition)

B = 5 (by substitution)

Therefore, the partial fraction decomposition is, $\dfrac{7x-1}{x^2+x-1}=\dfrac{5}{x+2}+\dfrac{2}{x-1}$

$$\frac{7x-1}{x^2+x-1}=\frac{5}{x+2}+\frac{2}{x-1}$$

2. $\frac{4x+6}{x^2-9}$

$$\frac{4x-6}{x^2-9}=\frac{A}{x-3}+\frac{B}{x+3}$$

$$4x+6=A(x+3)+B(x-3)$$

$$4x+6=Ax+3A+Bx-3B$$

$$4x+6=Ax+Bx+3A-3B$$

$$4x+6=(A+B)x+(3A-3B)$$

A+ B = 4
3A-3B = 6
3A+3 B = 12 (multiply equation 1 by 3)
<u>3A- 3B = 6</u>
6A = 18 (by subtraction)
A = 3
B = 1

Therefore, the partial fraction decomposition is, $\frac{4x-6}{x^2-9}=\frac{3}{x-3}+\frac{1}{x+3}$

3. $\frac{2x+6}{x^2+3x+2}$

$$\frac{2x+6}{x^2+3x+2}=\frac{A}{x+1}+\frac{B}{x+2}$$

$$2x+6=A(x+2)+B(x+1)$$

$$2x+6=Ax+2A+Bx+B$$

$$2x+6=Ax+Bx+2A+B$$

$$2x+6=(A+B)x+(2A+B)$$

A + B = 2
<u>2A + B = 6</u>
-A = -4

A = 4
B = -2

Therefore, the partial fraction decomposition is, $\dfrac{2x+6}{x^2+3x+2}=\dfrac{4}{x+1}-\dfrac{2}{x+2}$

4. $\dfrac{4x-3}{2x^2-3x+1}$

$$\frac{4x-3}{2x^2-3x+1}=\frac{A}{x-1}+\frac{B}{2x-1}$$

$$4x-3=A(2x-1)+B(x-1)$$

$$4x-3=2Ax-A+Bx-B$$

$$4x-3=2Ax+Bx-A-B$$

$$4x-3=(2A+B)x+(-A-B)$$

2A + B = 4
-A - B = - 3
A = 1

B = 2

Therefore, the partial fraction decomposition is, $\dfrac{4x-3}{2x^2-3x+1}=\dfrac{1}{x-1}+\dfrac{2}{2x-1}$

5. $\dfrac{(-2)x-19}{6x^2+7x-3}$

$$\frac{(-2)x-19}{6x^2+7x-3}=\frac{A}{3x-1}+\frac{B}{2x+3}$$

$$(-2)x+19=A(2x+3)+(3x-1)$$

$$(-2)x+19=2Ax+3A+3Bx-B$$

$$(-2)x+19=2Ax+3Bx+3A-B$$

$$(-2)x+19=(2A\cdot 3B)x+(3A-B)$$

2A + 3B = -2
3A – B = 19
2A + 3B = -2
<u>9A - 3B = 57</u>
11A = 55
A = 5
B = -4

Therefore, the partial fraction decomposition is, $\dfrac{(-2)x-19}{6x^2+7x-3}=\dfrac{5}{3x-1}-\dfrac{4}{2x+3}$

6. $\dfrac{-x-8}{2x^2-3x-2}$

$$\frac{-x-8}{2x^2-3x-2}=\frac{A}{x-2}+\frac{B}{2x-1}$$

$$-x-8=A(2x+1)+B(x-2)$$

$$-x-8=2Ax+A+Bx-2B$$

$$-x-8=2Ax+Bx+B-2B$$

$$-x-8=(2A+B)x+(B-2B)$$

2A +B = -1

A - 2B = -8

4A + 2B =-2 (multiply equation 1 by 2)

A - 2B =-8

5A = -10

A = -2

B = 3 (by substitution)

Therefore, the partial fraction decomposition is, $\frac{-x-8}{2x^2-3x-2}=\frac{3}{2x-1}-\frac{2}{x-2}$

SECTION 2

REPEATED QUADRATIC FACTORS

Examples in the use of rule (2).

The worked problems above are examples of the use of rule (1) where for every linear factor, ***(x-a)***, there exists a partial fraction in the form ***A/(x-a).*** The following problems will demonstrate the use of rule (2).

<u>**Example 3**</u>

Find the partial fraction decomposition of,

$$\frac{x+5}{(x+2)^2}$$

Here we need two partial fractions in the form ***A/(x-a)*** and ***B/(x-a)²***.

$$\frac{x+5}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$$

Now we follow steps (2) to (5) above.

$$(x+2)^2\left(\frac{x+5}{(x+2)^2}\right)=(x+2)^2\left(\frac{A}{x+2}\right)+(x+2)^2\cdot\left(\frac{B}{(x+2)^2}\right)$$

$$(x+5)=A(x+2)+B$$

$$x+5=(Ax+2A)+(B)$$

$$x+5=Ax+2A+B$$

$$x+5=(A)x+(2A+B)$$

$A \quad = 1$

$2A + B = 5$

B = 3 (by substitution)

Therefore, the partial fraction decomposition is, $\frac{x+5}{(x+2)^2}=\frac{1}{x+2}+\frac{3}{(x+2)^2}$

<u>Example 4</u>

Find the partial fraction decomposition of,

$$\frac{2x-2}{(x-3)^2}$$

$$\frac{2x-2}{(x-3)^2}=\frac{A}{x-3}+\frac{B}{(x-3)^2}$$

$$(x-3)^2\left(\frac{2x-2}{(x-3)^2}\right)=(x-3)^2\left(\frac{A}{x-3}\right)+(x-3)^2\left(\frac{B}{(x-3)^2}\right)$$

$$2x-2=A(x-3)+B$$

$$2x-2=Ax-3A+B$$

$$2x-2=(A)x-(3A-B)$$

$A = 2$
$-3A+B=-2$
$A=2$
$B=4$

Therefore, the partial fraction decomposition is, $\frac{2x-2}{(x-3)^2}=\frac{2}{x-3}+\frac{4}{(x-3)^2}$

Independent Practice 2

Find the partial fraction decomposition.

1. $\frac{3x-2}{(x-1)^2}$
2. $\frac{5x-6}{(x-2)^2}$

3. $\frac{7x^2-16x+3}{2x(x-1)^2}$

4. $\frac{2x^2+13x+36}{x(x+3)^2}$

5. $\frac{(-3)x^2+11x-18}{(x-2)(x+2)^2}$

6. $\frac{3x^2-4x+2}{(x-1)^3}$

Solutions to independent practice 2.

1 $\frac{3x-2}{(x-1)^2}$

$$\frac{3x-2}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}$$

$$3x-2=A(x-1)+B$$

$$3x-2=Ax-A+B$$

$$3x-2=(A)x-(A-B)$$

$A = 3$

$-A+B=-2$

$A=3$

$B=1$

Therefore, the partial fraction decomposition is, $\frac{3x-2}{(x-2)^2}=\frac{3}{x-1}+\frac{1}{(x-1)^2}$

2 $\frac{5x-6}{(x-2)^2}$

$$\frac{5x-6}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$5x-6=A(x-2)+B$$

$$5x-6=Ax-2A+B$$

$$5x-6=(A)x-(2A-B)$$

A = 5
-2A + B = -6
A = 5, B = 4

Therefore, the partial fraction decomposition is, $\frac{5x-6}{(x-2)^2}=\frac{5}{x-2}+\frac{4}{(x-2)^2}$

3 $\frac{7x-16x+3}{2x(x-1)^2}$

$$\frac{7x-16x+3}{2x(x-1)^2}=\frac{A}{2x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$7x^2-16x+3=A(x-1)^2+B(2x)(x-1)+2Cx$$

$$7x^2-16x+3=Ax^2-2Ax+A+2Bx^2-2Bx+2Cx$$

$$7x^2-16x+3=(A+2B)x^2-(2A+2B-C)x+(A)$$

A + 2B = 7
-2A- 2B + C = -16

A = *3*

A= *3*

B= *2*

C= *-3*

Therefore, the partial fraction decomposition is, $\frac{7x^2-16x+3}{2x(x-1)^2}=\frac{3}{2x}+\frac{2}{x-1}-\frac{3}{(x-2)^2}$

4 $\frac{2x^2+13x+36}{x(x+3)^2}$

$$\frac{2x^2+13x+36}{x(x+3)^2}=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$$

$$2x^2+13x+36=A(x+3)^2+Bx(x+3)+Cx$$

$$2x^2+13x+36=A(x^2+6x+9)+Bx^2+3Bx+Cx$$

$$2x^2+13x+36=Ax^2+6Ax+9A+Bx^2+3Bx+Cx$$

$$2x^2+13x+36=Ax^2+Bx^2+6Ax+3Bx+Cx+9A$$

$$2x^2+13x+36=(A+B)x^2+(6A+3B+C)x+9A$$

A +B = *2*

6A+ 3B + C = 13

9A = *36*

A = *4*

B = -2

C = -5

Therefore, the partial fraction decomposition is, $\frac{2x^2+13x+36}{x(x+3)^2}+\frac{4}{x}-\frac{2}{x+3}-\frac{5}{(x+3)^2}$

5 $\frac{(-3)x^2+11x-18}{(x-2)(x+2)^2}$

$$\frac{(-3)x^2+11x-18}{(x-2)(x+2)^2}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

$$(-3)x^2+11x-18=A(x+2)^2+B(x-2)(x+2)+C(x-2)$$

$$(-3)x^2+11x-18=A(x^2+4x+4)+B(x^2-4)+Cx-2C$$

$$(-3)x^2+11x-18=Ax^2+4Ax+4A+Bx^2-4B+Cx-2C$$

$$(-3)x^2+11x-13=Ax^2+Bx^2+4Ax+Cx+4A-4B-2C$$

$$(-3)x^2+11x-13=(A+B)x^2+(4A+C)x+(4A-4B-2C)$$

$$\begin{array}{lcl} A+B & & =-3 \\ 4A & +C & =11 \\ 4A+4B & -2C & =-18 \end{array}$$

$A=2$

$B=-5$

$C=3$

Therefore, the partial fraction decomposition is, $\frac{(-3)x^2+11x-18}{(x-2)(x+2)^2}=\frac{2}{x-2}-\frac{5}{x+2}+\frac{3}{(x+2)^2}$

6. $$\frac{3x^2-4x+2}{(x-1)^3}$$

$$\frac{3x^2-4x+2}{(x-1)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}$$

$$3x^2-4x+2=A(x-1)^2+B(x-1)+C$$

$$3x^2-4x+2=A(x^2-2x+1)+Bx-B+C$$

$$3x^2-4x+2=Ax^2-2Ax+A+Bx-B+C$$

$$3x^2-4x+2=Ax^2-2Ax+Bx+A-B+C$$

$$3x^2-4x+2=(A)x^2-(2A-B)x+(A-B+C)$$

$A = 3$

$-2A + B = -4$

$A - B + C = 2$

$A = 3$

$B = 2$

$C = 1$

Therefore, the partial fraction decomposition is, $$\frac{3x^2-4x+2}{(x-3)^3}=\frac{3}{x-1}+\frac{2}{(x-1)^2}+\frac{1}{(x-1)^3}$$

<u>SECTION 3</u>

REPEATED QUADRATIC FACTORS

Examples in the use of rule (3).

In this section we will use rule (3) to find the partial fraction decomposition of rational expressions. Here we will assign numerators, ***(Ax+B)***, ***(Cx+D)*** etc. to any rational expression with an irreducible quadratic denominator.

Example 5

Find the partial fraction decomposition of,

$$\frac{3x^2+2x+5}{x^3+x^2+x+1}$$

First factor the denominator then use constants ***A***, ***B*** and ***C*** as numerators. The factors are one linear, (***x+1***) and one irreducible quadratic, (x^2 ***+1***). We assign ***A*** as the numerator of the linear, and (***Bx*** + ***C***) as the numerator of the quadratic.

$$\frac{3x^2+2x+5}{x^3+x^2+x+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

Multiplying both sides by the least common denominator,

$$(x^3+x^2+x+1)\left(\frac{3x^2+2x+5}{x^3+x^2+x+1}\right)=(x^3+x^2+x+1)\left(\frac{A}{x+1}\right)+(x^3+x^2+x+1)\left(\frac{Bx+C}{x^2+1}\right)$$

We obtain.

$$3x^2+2x+5=A(x^2+1)+(Bx+C)(X+1)$$

Apply the distributive property to the left side.

$$3x^2+2x+5=Ax^2+A+Bx^2+Bx+Cx+C$$

Rearrange in descending powers of ***x***.

$$3x^2+2x+5=Ax^2+Bx^2+Bx+Cx+A+C$$

Factor out the powers of ***x***.

$$3x^2+2x+5=(A+B)x^2+(B+C)x+(A+C)$$

Set the coefficients equal.

$A + B \quad = 3$

$\quad B + C = 2$

$A \quad + C + 5$

Solving,

$A+B \quad = 3$

$\underline{B + C = 2}$ (By subtraction)

$A + C = 1$

$\underline{A + C = 5}$ (By addition)

$2A \quad + 6$

A = 3

B = 0 (By substitution)

C = 2

Therefore, the partial fraction decomposition is, $\frac{3x^2+2x+5}{x^3+x^2+x+1} = \frac{3}{x+1} + \frac{2}{x^2+1}$

Example 6

Find the partial fraction decomposition of,

$$\frac{4x^2+4x-6}{2x^3-2x^2-x+1}$$

$$\frac{4x^2+4x-6}{2x^3-2x^2-x+1} = \frac{A}{x-1} + \frac{Bx+C}{2x^2-1}$$

$$4x^2+4x-6 = A(2x^2-1) + (Bx+C)(x-1)$$

$$4x^2+4x-6=2Ax^2-A+Bx^2-Bx+Cx-C$$

$$4x^2+4x-6=2Ax^2+Bx^2-Bx+Cx-A-C$$

$$4x^2+4x-6=(2A+B)x^2-(B-C)x-(A+C)$$

$2A + B \quad = 4$
$\quad -B + C = 4$
$-A \quad -C = -6$
$A = 2$
$B = 0$
$C = 4$

Therefore, the partial fraction decomposition is, $\dfrac{4x^2+4x-6}{2x^3-2x^2-x+1}=\dfrac{2}{x-1}+\dfrac{4}{2x^2-1}$

Example 7

Find the partial fraction decomposition of,

$$\frac{3x^2-3x-13}{3x^3+15x^2+2x+10}$$

$$\frac{3x^2-3x-13}{3x^3+15x^2+2x+10}=\frac{A}{x+5}+\frac{Bx+C}{3x^2+2}$$

$$3x^2-3x-13=A(3x^2+2)+(Bx+C)(x+5)$$

$$3x^2-3x-13=3Ax^2+2A+Bx^2+5Bx+Cx+5C$$

$$3x^2-3x-13=3Ax^2+Bx^2+5Bx+Cx+2A+5C$$

$$3x^2-3x-13=(3A+B)x^2+(5B+C)x+(2A+5C)$$

$3A + B = 3$

$5B + C = -3$
$2A + \quad 5C = -13$
$A = 1$
$B = 0$
$C = -3$

Therefore, the partial fraction decomposition is, $\frac{3x^2 - 3x - 13}{3x^3 + 15x^2 + 2x + 10} = \frac{1}{x+5} - \frac{3}{3x^2+2}$

Example 8

Find the partial fraction decomposition of,

$$\frac{3x^3 - 2x^2 + 2x - 5}{(x^2+2)^2}$$

$$\frac{3x^3 - 2x^2 + 2x - 5}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$$

$$3x^3 - 2x^2 + 2x - 5 = (Ax+B)(x^2+2) + Cx + D$$

$$3x^3 - 2x^2 + 2x - 5 = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$

$$3x^3 - 2x^2 + 2x - 5 = Ax^3 + Bx^2 + 2Ax + Cx + 2B + D$$

$$3x^3 - 2x^2 + 2x - 5 = (A)x^3 + (B)x^2 + (2A + C)x + (2B + D)$$

$A \quad = 3$
$B \quad = -2$
$2A + C = 2$
$2B + D = -5$
$A = 3$
$B = -2$
$C = -4$

D = -1

Therefore, the partial fraction decomposition is, $\dfrac{3x^3-2x^2+2x-5}{(x^2+2)^2}=\dfrac{3x-2}{x^2+2}-\dfrac{4x+1}{(x^2+2)^2}$

<u>Independent Practice 3</u>

Find the partial fraction decomposition.

1. $\dfrac{4x^2-5x-17}{2x^3+6x^2-x-3}$
2. $\dfrac{8x^2+3x}{2x^3+6x^2+3x+9}$
3. $\dfrac{2x^2+9x+1}{x^3+4x^2+2x-1}$
4. $\dfrac{4x^2+6x}{x^3-2x^2-2x-3}$
5. $\dfrac{5x^2-17x+23}{x^3-7x^2+15x-12}$
6. $\dfrac{3x^3-4x^2-4x-2}{(x^2-x-1)^2}$

Solutions to independent practice 3.

1. $\dfrac{4x^2-5x-17}{2x^3+6x^2-x-3}$

$$\frac{4x^2-5x-17}{2x^3+6x^2-x-3}=\frac{A}{x+3}+\frac{Bx+C}{2x^2-1}$$

$$4x^2-5x-17=A(2x^2-1)+(Bx+C)(x+3)$$

$$4x^2-5x-17=2Ax^2-A+Bx^2+3Bx+Cx+3C$$

$$4x^2-5x-17=2Ax^2+Bx^2+3Bx+Cx-A+3C$$

$$4x^2-5x-17=(2A+B)x^2+(3B+C)x-(A-3C)$$

$2A + B \quad = 4$

$\quad 3B + C = -5$

$-A \quad +3C = -17$

$A = 2$

$B = 0$

$C = -5$

Therefore, the partial fraction decomposition is, $\dfrac{4x^2-5x-17}{2x^3+6x^2-x-3}=\dfrac{2}{x+3}-\dfrac{5}{2x^2-1}$

2. $\dfrac{8x^2-3x}{2x^3+6x^2+3x+9}$

$$\frac{8x^2-3x}{2x^3+6x^2+3x+9}=\frac{A}{x+3}+\frac{Bx+C}{2x^2+3}$$

$$8x^2+3x=A(2x^2+3)+(Bx+C)(x+3)$$

$$8x^2+3x=2Ax^2+3A+Bx^2+3Bx+Cx+3C$$

$$8x^2+3x=2Ax^2+Bx^2+3Bx+Cx+3A+3C$$

$$8x^2+3x=(2A+B)x^2+(3B+C)x+(3A+3C)$$

2A + B = 8
3B + C = 3
3A 3C = 0
A = 3
B = 2
C = -3

Therefore, the partial fraction decomposition is, $\dfrac{8x^2-3x}{2x^3+6x^2+3x+9}=\dfrac{3}{x+3}+\dfrac{2x-3}{2x^2+3}$

3. $\dfrac{2x^2+9x+1}{x^3+4x^2+2x-1}$

$$\frac{2x^2+9x+1}{x^3+4x^2+2x-1}=\frac{A}{x+1}+\frac{Bx+C}{x^2+3x-1}$$

$$2x^2+9x+1=A(x^2+3x-1)+(Bx+C)(x+1)$$

$$2x^2+9x+1=Ax^2+3Ax-A+Bx^2+Bx+Cx+C$$

$$2x^2+9x+1=Ax^2+Bx^2+3Ax+Bx+Cx-A+C$$

$$2x^2+9x+1=(A+B)x^2+(3A+B+C)x-(A-C)$$

A + B = 2
3A + B + C = 9

-A   + C = 1

A = 2

B = 0

C = 3

Therefore, the partial fraction decomposition is, $\dfrac{2x^2+9x+1}{x^3+4x^2+2x-1}=\dfrac{2}{x+1}+\dfrac{3}{x^2+3x-1}$

4. $\dfrac{4x^2+6x}{x^3-2x^2-2x-3}$

$$\frac{4x^2+6x}{x^3-2x^2-2x-3}=\frac{A}{x-2}+\frac{Bx+C}{x^2+x+1}$$

$$4x^2+6x=A(x^2+x+1)+(Bx+C)(x-2)$$

$$4x^2+6x=Ax^2+Ax+A+Bx^2-2Bx+Cx-2C$$

$$4x^2+6x=Ax^2+Bx^2+Ax-2Bx+Cx+A-2C$$

$$4x^2+6x=(A+B)x^2+(A-2B+C)x+(A-2C)$$

A + B   = 4

A - 2B +C   = 6

A   - 2C   = 0

A = 4

B = 0, C = 2

Therefore, the partial fraction decomposition is, $\dfrac{4x^2+6x}{x^3-2x^2-2x-3}=\dfrac{4}{x-2}+\dfrac{2}{x^2+x+1}$

5. $$\frac{5x^2-17x+23}{x^3-7x^2+15x-12}$$

$$\frac{5x^2-17x+23}{x^3-7x^2+15x-12}=\frac{A}{x-4}+\frac{Bx+C}{x^2-3x+3}$$

$$5x^2-17x+23=A(x^2-3x+3)+(Bx+C)(x-4)$$

$$5x^2-17x+23=Ax^2-3Ax+3A+Bx^2-4Bx+Cx-4C$$

$$5x^2-17x+23=Ax^2+Bx^2-3Ax-4Bx+Cx+3A-4C$$

$$5x^2-17x+23=(A+B)x^2-(3A+4B-C)x+(3A-4C)$$

$$\begin{aligned} A + B \quad &= 5 \\ -3A - 4B + C &= -17 \\ 3A \quad - 4C &= 23 \end{aligned}$$

$A = 5$
$B = 0$
$C = -2$

Therefore, the partial fraction decomposition is, $\dfrac{5x^2-17x+23}{x^3-7x^2+15x-12}=\dfrac{5}{x-4}-\dfrac{2}{x^2-3x+3}$

6. $$\frac{3x^3-4x^2-4x-2}{(x^2-x-1)^2}$$

$$\frac{3x^3-4x^2-4x-2}{(x^2-x-1)^2}=\frac{Ax+B}{x^2-x-1}+\frac{Cx+D}{(x^2-x-1)^2}$$

$$3x^3-4x^2-4x-2=(Ax+B)(x^2-x-1)+Cx+D$$

$$3x^3 - 4x^2 - 4x - 2 = Ax^3 - Ax^2 - Ax + Bx^2 - Bx - B + Cx + D$$

$$3x^3 - 4x^2 - 4x - 2 = Ax^3 - Ax^2 + Bx^2 - Ax - Bx + Cx - B + D$$

$$3x^3 - 4x^2 - 4x - 2 = (A)x^3 - (A - B)x^2 - (A + B + C)x - (B - D)$$

$$\begin{aligned} A &= 3 \\ -A + B &= -4 \\ -A - B + C &= -4 \\ -B + D &= 2 \end{aligned}$$

$$A = 3$$
$$B = -1$$
$$C = -2,\ D = -3$$

Therefore, the partial fraction decomposition is, $\dfrac{3x^3 - 4x^2 - 4x - 2}{(x^2 - x - 1)^2} = \dfrac{3x - 1}{x^2 - x - 1} - \dfrac{2x + 3}{(x^2 - x - 1)^2}$

SECTION 4

THE DEGREE OF THE NUMERATOR IS GREATER

Examples in the use of rule (4).

Example 9

Find the partial fraction decomposition of,

$$\frac{2x^3 - 7x^2 + 10x + 24}{2X^2 - 3x - 9}$$

Since the degree of the numerator is greater that of the denominator, we must divide the numerator by the denominator. The result is,

$$\frac{2x^3-7x^2+10x+24}{2X^2-3x-9}=x-2+\frac{13x+6}{2x^2-3x-9}$$

Decomposing the fraction, we get

$$\frac{13x+6}{2X^2-3x-9}=\frac{A}{x-3}+\frac{B}{2x+3}$$

Now we follow steps (2) to (5) as in example (1) above.

$$13x+6=A(2x+3)+B(x-3)$$

$$13x+6=2Ax+3A+Bx-3B$$

$$13x+6=2Ax+Bx+3A-3B$$

$$13x+6=(2A+B)x+(3A-3B)$$

2A + B = 13
3A + 3B = 6
A= 5
B= 3

Therefore, the partial fraction decomposition is, $\frac{2x^3-7x^2+10x+24}{2X^2-3x-9}=x-2+\frac{5}{x-3}+\frac{3}{2x+3}$

<u>Example 10</u>

Find the partial fraction decomposition of,

$$\frac{4x^3-16x^2+17x-9}{2x^2-9x+9}$$

$$\frac{4x^3-16x^2+17x-9}{2x^2-9x+9}=2x+1+\frac{8x-18}{2x^2-9x+9}$$ (dividing)

$$\frac{8x-18}{2x^2-9x+9}=\frac{A}{2x-3}+\frac{B}{x-3}$$ (follow steps 1-5)

$$8x-18=A(x-3)+B(2x-3)$$

$$8x-18=Ax-3A+2Bx-3B$$

$$8x-18=Ax+2Bx-3A-3B$$

$$8x-18=(A+2B)x-(3A+3B)$$

$A + 2B = 8$
$-3A - 3B = -18$
$A = 4$
$B = 2$

Therefore, the partial fraction decomposition is, $$\frac{4x^3-16x^2+17x-9}{2x^2-9x+9}=2x+1+\frac{4}{2x-3}+\frac{2}{x-3}$$

<u>Example 11</u>

Find the partial fraction decomposition of,

$$\frac{x^3+x^2-5x+1}{x^2-x-2}$$

$$\frac{x^3+x^2-5x+1}{x^2-x-2}=x+2+\frac{-x+5}{x^2-x-2}$$

$$\frac{-x+5}{x^2-x-2}=\frac{A}{x-2}+\frac{B}{x+1}$$

$$-x+5=A(x+1)+B(x-2)$$

$$-x+5=Ax+A+Bx-2B$$

$$-x+5=Ax+Bx+A-2B$$

$$-x+5=(A+B)x+(A-2B)$$

A + B = 1
A - 2B = 5

A = 1
B = -2

Therefore, the partial fraction decomposition is, $\dfrac{x^3+x^2-5x+1}{x^2-x-2}=x+2+\dfrac{1}{x-2}-\dfrac{2}{x+1}$

<u>Independent Practice 4</u>

Find the partial fraction decomposition.

1. $\dfrac{3x^3+8x^2+7x+5}{x^2+3x+2}$

2. $\dfrac{2x^3+3x^2-11x-10}{x^2+2x-3}$

3. $$\frac{12x^3+17x^2+10x+3}{3x^2+5x+2}$$

4. $$\frac{4x^3-8x^2+9x+2}{2x^2-x-1}$$

5. $$\frac{3x^3-x^2-9x}{x^2-x-2}$$

6. $$\frac{x^3-3x-4}{x^2-3x+2}$$

Solutions to independent practice 4.

Find the partial fraction decomposition of,

1. $$\frac{3x^3+8x^2+7x+5}{x^2+3x+2}$$

$$\frac{3x^3+8x^2+7x+5}{x^2+3x+2}=3x-1+\frac{4x+7}{x^2+3x+2}$$

$$\frac{4x+7}{x^2+3x+2}=\frac{A}{x+1}+\frac{B}{x+2}$$

$$4x+7=A(x+2)+B(x+1)$$

$$4x+7=Ax+2A+Bx+B$$

$$4x+7=Ax+Bx+2A+B$$

$$4x+7=(A+B)x+(2A+B)$$

$A+B = 4$
$2A+B = 7$
$A = 3$
$B = 1$

Therefore, the partial fraction decomposition is, $\dfrac{3x^3+8x^2+7x+5}{x^2+3x+2}=3x-1+\dfrac{3}{x+1}+\dfrac{1}{x+2}$

Find the partial fraction decomposition of,

2. $\dfrac{2x^3+3x^2-11x-10}{x^2+2x-3}$

$$\frac{2x^3+3x^2-11x-10}{x^2+2x-3}=2x-1+\frac{(-3)x-13}{x^2+2x-3}$$

$$\frac{(-3)x-13}{x^2+2x-3}=\frac{A}{x+3}+\frac{B}{x-1}$$

$$(-3)x-13=A(x-1)+B(x+3)$$

$$(-3)x-13=Ax-A+Bx+3B$$

$$(-3)x-13=Ax+Bx-A+3B$$

$$(-3)x-13=(A+B)x-(A-3B)$$

$A + B = -3$
$-A+3B = -13$
$A = 1$

B = -4

Therefore, the partial fraction decomposition is, $$\frac{2x^3+3x^2-11x-10}{x^2+2x-3}=2x-1+\frac{1}{x+3}-\frac{4}{x-1}$$

Find the partial fraction decomposition of,

3. $$\frac{12x^3+17x^2+10x+3}{3x^2+5x+2}$$

$$\frac{12x^3+17x^2+10x+3}{3x^2+5x+2}=4x-1+\frac{7x+5}{3x^2+5x+2}$$

$$\frac{7x+5}{3x^2+5x+2}=\frac{A}{x+1}+\frac{B}{3x+2}$$

$$7x+5=A(3x+2)+B(x+1)$$

$$7x+5=3Ax+2A+Bx+B$$

$$7x+5=3Ax+Bx+2A+B$$

$$7x+5=(3A+B)x+(2A+B)$$

3A + B = 7
2A + B =5
A = 2
B =1

Therefore, the partial fraction decomposition is, $$\frac{12x^3+17x^2+10x+3}{3x^2+5x+2}=4x-1+\frac{2}{+x+1}+\frac{1}{3x+2}$$

Find the partial fraction decomposition of,

4. $$\frac{4x^3-8x^2+9x+2}{2x^2-x-1}$$

$$\frac{4x^3-8x^2+9x+2}{2x^2-x-1}=2x-3+\frac{8x+1}{2x^2-x-1}$$

$$\frac{8x+1}{2x^2-x-1}=\frac{A}{2x+1}+\frac{B}{x-1}$$

$$8x+1=A(x-1)+B(2x+1)$$

$$8x+1=Ax-A+2Bx+B$$

$$8x+1=Ax+2Bx-A+B$$

$$8x+1=(A+2B)x-(A-B)$$

A + 2B = 8
-A + B = 1
A = 2
B = 3

Therefore, the partial fraction decomposition is, $\frac{4x^3-8x^2+9x+2}{2x^2-x-1}=2x-3+\frac{2}{2x+1}+\frac{3}{x-1}$

Find the partial fraction decomposition of,

5. $$\frac{3x^3-x^2-9x}{x^2-x-2}$$

$$\frac{3x^3-x^2-9x}{x^2-x-2}=3x+2+\frac{-x+4}{x^2-x-2}$$

$$\frac{-x+4}{x^2-x-2}=\frac{A}{x-1}+\frac{B}{x+2}$$

$$-x+4=A\,(x+2)+B\,(x-1)$$

$$-x+4=A\,x+2\,A+B\,x-B$$

$$-x+4=A\,x++B\,x+2\,A-B$$

$$-x+4=(A++B)\,x+(2\,A-B)$$

A + B = -1
2A - B = 4
A = 1
B = -2

Therefore, the partial fraction decomposition is, $\frac{3\,x^3-x^2-9\,x}{x^2-x-2}=3\,x+2+\frac{1}{x-1}-\frac{2}{x+2}$

Find the partial fraction decomposition of,

6. $\frac{x^3-3\,x-4}{x^2-3\,x+2}$

$$\frac{x^3-3\,x-4}{x^2-3\,x+2}=x+3+\frac{4\,x-10}{x^2-3\,x+2}$$

$$\frac{4\,x-10}{x^2-3\,x+2}=\frac{A}{x-1}+\frac{B}{x-2}$$

$$4\,x-10=A\,(x-2)+B\,(x-1)$$

$$4\,x-10=A\,x-2\,A+B\,x-B$$

$$4x - 10 = Ax + Bx - 2A - B$$

$$4x - 10 = (A + B)x - (2A + B)$$

$A + B = 4$
$-2A - B = -10$
$A = 6$
$B = -2$

Therefore, the partial fraction decomposition is, $\frac{x^3 - 3x - 4}{x^2 - 3x + 2} = x + 3 + \frac{6}{x - 1} - \frac{2}{x - 2}$